OPEN PROBLEMS – OBERWOLFACH 2022

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1. Well-linked sets in directed graphs (Ken-ichi Kawarabayashi)

Two sets S and T with $|S| = |T|$ are (S,T)-*well-linked* in a directed graph if for any $A \subseteq S$ and $B \subseteq T$ with $|A| = |B|$, there are $|A| = |B|$ disjoint paths (linkage) from A to B. Note: S, T may not be disjoint. Its order is $|S| + |T|$

Motivation:

- Sparsest directed cut (for general graphs, no good approximation, but for planar graphs, $O(\log^3 n)$ approx. by Sidiropoulos and KK in FOCS'21).

  More precisely, directed graph decomposition based on sparsest cuts. (For undirected graphs, this leads to a well-linked set decomposition, and this method leads to the polynomial grid theorem).
- Disjoint paths problem (in planar graphs) $S = \{s_1, s_2, s_3 \dots, s_k\}$, $T = \{t_1, t_2, t_3, \dots, t_k\}$. See below.
- A generalization of "well-linked" set (this (S,T)-well linked set can be defined even for DAG).

  Indeed, we are interested in DAG when there is no such a (S,T)-well linked set for any S, T. Maybe in some cases, some algorithmic questions can be faster?

Some problems:

- Okamura-Seymour for directed planar graphs:

  More precisely, for a directed planar graph G with the outer boundary C, if all vertices in $S \cup T$ are in C, and S, T satisfy a $(S.T)$-well-linked set, there are paths P_i with source node s_i and terminal node t_i, for $i = 1, \dots, k$, such that each vertex in G is used in at most two (or any constant number) of the paths.
- Polynomial acyclic grid theorem:

  If (S,T) is well-linked of order $f(k)$, there is an acyclic grid W of order k (i.e., W consists of two linkages X, Y of order k, such that X is from top to bottom, and Y is from left to right), as a minor.

  Moreover f is a polynomial function of k

  We are actually interested in a more relaxed form: G contains either W or biclique of order k as a minor. We are even interested in the case when G is a DAG (or G has no k disjoint cycles).

  If this kind of a form is true, we have a very good chance to show the polynomial bound for Erdős-Posa for directed disjoint cycles (i.e., a polynomial version of Younger's conjecture)

2. Linear rank-width of graphs excluding some tree as a vertex-minor (O-joung Kwon)

For a linear ordering $L = v_1, v_2, \dots, v_n$ of vertices of a graph G, the *width* of L is defined as the maximum rank of the matrices $A(G)[\{v_1 \dots v_i\}, \{v_{i+1}, \dots v_n\}]$, where $A(G)$ denotes the adjacency matrix, and the rank is computed over the binary field. The *linear rank-width of* G is the minimum width over all linear orderings of G.

Local complementation at a vertex v is the operation that replaces the subgraph induced by $N(v)$ with its complement. H is a *vertex-minor* of a graph G if H can be obtained from G by local complementations and vertex deletions.

Problem: For every tree T, does the class of graphs having no T vertex-minor have bounded linear rank-width?

Related papers:

- The grid theorem for vertex-minors JCTB accepted (Geelen, Kwon, McCarty, and Wollan)
- Obstructions to bounded rank-depth and shrub-depth JCTB 2021 (Kwon, McCarty, Oum, and Wollan)
- Tree pivot-minors and linear rank-width SIDMA 2021 (Dabrowski, Dross, Jeong, Kanté, Oum, and Paulusma)

3. Twin-width of graphs (Sang-il Oum)

Question. What is the maximum twin-width of an n-vertex graphs?

It was proved in [Ahn, Kevin Hendrey, Donggyu Kim, Sang-il Oum, Bounds for the Twin-width of Graphs, arXiv:2110.03957] that Payley graphs have twin-width equal to $\frac{n-1}{2}$. The paper also contains an upper bound for general graphs (of order $\frac{n}{2} + O(\sqrt{n \log n})$).

4. Treewidth of hereditary classes (Nicolas Trotignon)

The following conjecture was made by several people:

Conjecture: For every integer ℓ, there exists $C_\ell > 0$ such that if a graph G contains none of the following as an induced subgraph:

- subdivision of an $\ell \times \ell$ wall
- line graph of a subdivision of an $\ell \times \ell$ wall
- $K_{\ell,\ell}$
- K_ℓ

then, treewidth$(G) \leqslant C_\ell \log |V(G)|$.

Remarks:

- It would be a nice "induced subgraph" version of the celebrated Robertson and Seymour grid theorem, and maybe too much to believe. So, particular cases would be interesting. Also, trying to disprove it would be interesting.
- A weaker statement is proved by Tara Abrishami, Maria Chudnovsky, Sepehr Hajebi, and Sophie Spirkl in *Induced subgraphs and tree-decompositions III. Three-path-configurations and logarithmic tree-width*, available in arxiv 2109.01310.
- The logarithm in the conclusion is needed, as shown by a construction of Ni Luh Dewi Sintiari and Nicolas Trotignon described in *(Theta, triangle)-free and (even hole, K4)-free graphs. Part 1 : Layered wheels*, available in arxiv 1906.10998.

Variant proposed by S. Thomassé: under the same conditions, but with the $t \times t$-wall replaced by any fixed cubic graph H, the graph G has bounded twin-width.

5. Hitting all maximum independent sets (Noga Alon)

For a graph $G = (V, E)$ on n vertices let $\alpha(G)$ denote its independence number, and let $h(G)$ denote the minimum cardinality of a set S of vertices that intersects all maximum independent sets of G (that is, $\alpha(G - S) < \alpha(G)$).

Conjecture (Bollobás, Erdős and Tuza, 1991): If $\alpha(G) = \Omega(n)$ then $h(G) = o(n)$.

A relaxed conjecture: If $\chi(G) = O(1)$ then $h(G) = o(n)$. (Open even for $\chi(G) = 3$.)

Remarks:

- Hajnal (1965): If $\alpha(G) > n/2$ then $h(G) = 1$.
- There are graphs $G = G_n$ on n vertices with $\alpha(G) > n/4$, $\chi(G) \leqslant 8$ and $h(G) > \sqrt{n}/2$, and there are graphs $G = G_n$ on n vertices with $\alpha(G) = (1/2 - o(1))n$ and $h(G) > (\log n)^{0.999}$. This settles questions of Friedgut, Kalai and Kindler, and of Dong and Wu.
- If G is regular and $\alpha(G) > 0.250001n$ then $h(G) \leqslant O(\sqrt{n \log n})$. In particular this holds for regular 3-colorable graphs.

6. Beyond Hadwiger in F-free graphs (Matija Bucić)

Question (B., Fox and Sudakov). For which graphs F does the following hold:
G does not contain F as a subgraph $\implies$ $\exists$ a clique minor of size $(\chi(G))^{1+c}$ for $c = c(F) > 0$?

- Kuhn-Osthus, 2005: true if F is bipartite and ask the question for $F = K_s$.
- Dvořák and Kawarabayashi, 2017: not true if F contains a triangle.
- Delcourt and Postle 2021: showed there is a linear sized clique minor $\forall F$
- B., Fox and Sudakov 2021: true $\forall F$ with Hall ratio in place of χ.

7. Tree decompositions and neighborhoods of induced paths (Maria Chudnovsky)

Let us say that a set $X \subseteq V(G)$ is *nice* if there is an induced path P in X such that every vertex of $X \setminus V(P)$ has a neighbor in P.

Let c be an integer. Let us say that X is *c-super nice* if X is nice an in addition for every $p \in V(P)$ the neighborhood of p has at most c connected components.

Let t be an integer and let G be a graph with no induced path of length t. It is known that G has a tree decomposition where every bag is nice (I believe we do not even need to assume P_t-free). Does that exist a function f such that G has a tree decomposition where every bag is $f(t)$-super nice?

Probably not, but I don't have a counterexample.

Is there a counterexample of min degree three, where every two non-adjacent vertices have incomparable neighborhoods, and no neighborhood of a vertex is connected?

This is related the algorithmic question of 3-coloring P_t-free graphs.

8. Grids and connectivity in digraphs (Stephan Kreutzer)

Societies. A *society in a digraph* is pair (G, Ω) where G is a digraph and Ω is a cyclic ordering of some set $\Omega(G) \subseteq V(G)$.

A *cross* in (G, Ω) is a pair P_1, P_2 of disjoint directed paths such that the endpoints of P_i are s_i, t_i and s_1, s_2, t_1, t_2 occur in Ω in this order.

Question. If (G, Ω) is a cross-free society, then is it true that there always is a planar digraph H with $\Omega(G) \subseteq V(H)$ such that H has an embedding into the plane with the vertices of $\Omega(H)$ appearing in the outer face in the order specified by Ω such that H allows for exactly the same connectivity between vertices of $\Omega(G)$ as G?

What should be true. We can replace $G \setminus \Omega(G)$ by a suitable grid to get at least the same connectivity as in G but we may get more.

9. Compressibility of acyclic digraphs (Andrzej Grzesik)

For an acyclic oriented graph F define the *compressibility* of F as the smallest integer k such that F is homomorphic to any tournament on k vertices.

One can observe that the compressibility of F determines the asymptotic answer to the Turán problem asking for the maximum number of edges in a graph not containing F as a subgraph, because an oriented equivalent of Erdős-Stone theorem holds with the compressibility instead of the chromatic number.

Question: For what graphs F the compressibility of F is linear/polynomial in the length of the longest directed path in F?

It is known to be linear for powers of paths and orientations of trees and cycles, polynomial for 2-outdegenerated graphs, and exponential for transitive tournaments.

10. Does Erdős-Posá hold for vertex minors? (Paul Wollan)

Conjecture: For every circle graph H, $\exists f$ such that $\forall k$ and G, either

- G has kH as a vertex-minor, or
- $\exists$ a *rank* $f(k)$ *pertubation* G^* of G such that G^* has no H vertex minor.

Remarks:

- The role of circle graphs in vertex minor structure is the analog of the role of planar graphs in graph minor structure
- Grid theorem for vertex minors says it suffices to prove that in a graph of bounded rank width, either we have
  - kH vertex minor, or
  - a bounded rank perturbation of G has no H vertex minor

11. Coloring of planar graphs with vertex pairings (Johannes Carmesin)

A *pairing* of a graph is a partition of its vertex set into sets of size two of non-adjacent vertices.

Question (C, Kurkofka, Mihaylov, Nevinson) Given a planar graph G with a pairing, can G be coloured with 11 colours such that paired vertices receive the same colour?

Remarks:

- originated from trying to extend the 4-colour theorem to 3D;
- **The answer to this question is 'no'**. This was pointed out by Noga Alon as well as Michal Pilipczuk and Lukasz Bozyk, see `https://www.jstor.org/stable/24966248?seq=1#metadata_info_tab_contents` and `https://mathworld.wolfram.com/EmpireProblem.html`
- upper bound of 12. This bound is tight, see the above references

12. 2-well-quasi-order of planar graphs (Nathan Bowler)

Question: Are planar graphs *2-well-quasi-ordered* under the minor relation?

That is, can we rule out the existence of a family of planar graphs $(G_{ij})_{i<j\in\mathbb{N}}$ such that there are no $i<j<k\in\mathbb{N}$ for which G_{ij} is a minor of G_{jk}?

13. An inequality for the symmetric group (Bhargav Narayanan)

Let $w : E(K_n) \to \mathbb{R}_{\geqslant 0}$ be any non-negative weighting of the edges of the complete graph on $[n]$ vertices. Given w, we associate two quantities to any permutation $\pi \in S_n$. First, the order-weight of π is given by

$$\mathrm{ord}(\pi) = \prod_{i=1}^{n-1} w(\pi(i), \pi(i+1));$$

this comes from looking at π as an 'ordering' of $[n]$, and then multiplying the weights on the edges of the Hamilton path corresponding to π. Second, the cycle-weight of π is given by

$$\mathrm{cyc}(\pi) = \prod_{i=1}^{n} w(i, \pi(i)),$$

where $w(j, j)$ is taken to be 1 for all $j \in [n]$; this comes from looking at π as a product of cycles, and then multiplying the weights on the edges in the cycle decomposition of π (with multiplicity, and with fixed-points contributing weight 1).

Here is a rather intriguing conjectural inequality: for all w as above, we have

$$\sum_{\pi \in S_n} \mathrm{cyc}(\pi) \geqslant \sum_{\pi \in S_n} \mathrm{ord}(\pi),$$

with equality only holding for w identically 1 on each edge.

For $n = 2$ with a single weight $x \geqslant 0$ on K_2, the inequality reads $1 + x^2 \geqslant 2x$, which is trivially true, and for $n = 3$ with weights $x, y, z \geqslant 0$ on the edges of K_3, the inequality reads $1 + x^2 + y^2 + z^2 + 2xyz \geqslant 2xy + 2yz + 2zx$, which can be verified with a little effort. My computer has verified the claim for $n = 4$, but I know of no nice proof.

This came from some joint work with Lisa Sauermann on counting spanning trees where we just wanted this fact for 0/1-valued w, i.e., graphs. In this special case, the inequality says that the number of Hamilton paths in any graph G (counted twice, one for each orientation) is at most the number of permutations of the vertex set where each vertex is sent to either itself or one of its neighbours. I do not know how to prove this in general either.

I do know the conjecture to be true when all weights are $\geqslant 1$, but this is rather simple. When all the weights are equal, the inequality follows from Jensen's inequality applied to the random variable tracking the number of fixed points of a permutation sampled uniformly from S_n.

14. Geometric reconstruction (Alex Scott)

Let S be a set of n points in $\mathbb{R}^d$. The *k-deck* of S is the multiset of all k-point subsets of S, given up to isometry. For example, the 2-deck of S is equivalent to knowing how many times each distance occurs in S. We say that a set S is *reconstructible from its k-deck* if every set with the same k-deck as S is isometric to S.

How large does k need to be so that every set of n points is reconstructible from its k-deck?

In one dimension, it is not hard to see that $k = 4$ is enough (i.e. every finite set of $\mathbb{R}$ is reconstructible from its 4-deck). But in two dimensions, the problem is more difficult. [N. Alon, Y. Caro, I. Krasikov and Y. Roditty, Combinatorial reconstruction problems, *J. Combin. Theory Ser. B* **47** (1989), 153–161] raised the question, and showed that every set of n points in
$mathbbR^2$ can be reconstructed from its $(\log_2 n + 1)$-deck. [L. Pebody, A. J. Radcliffe and A.

D. Scott, All finite subsets of the plane are 18-reconstructible, *SIAM J. Discrete Math.* **16** (2003), 262–275] showed that there is a constant k that will do for all finite sets (in fact $k = 36$ is enough).

In three or more dimensions, much less is known. The arguments of Alon, Caro, Krasikov and Roditty show that logarithmic size is enough, but there is no non-constant lower bound. With Jamie Radcliffe, I conjecture the following.

Conjecture: There is some $k \in \mathbb{N}$ such that every finite subset of $\mathbb{R}^3$ is determined up to isometry by its k-deck.

15. A variant of the Erdős–Faber–Lovász conjecture (Tom Kelly)

The Erdős–Faber–Lovász conjecture is the following: *If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^n G_i) \leqslant n$.*

In joint work with Dong Yeap Kang, Daniela Kühn, Abhishek Methuku, and Deryk Osthus from last year, we proved this conjecture for all sufficiently large n. There are still several variations and possible generalizations that remain open. One such example is the following:

Problem. *Let $G_1, \ldots, G_n$ be graphs, each of chromatic number at most $n-1$, such that every pair shares at most one vertex. What is the largest possible chromatic number of $\chi(\bigcup_{i=1}^n G_i)$?*

Erdős asked a related problem in 1981 that turned out to be trivial, but this is probably what he really wanted to ask.

In joint work with Daniela Kühn and Deryk Osthus, we proved an upper bound of $2n-3$. It is possible that the answer is simply n, which would actually imply the Erdős–Faber–Lovász conjecture.

16. Structure and coloring of 3-connected graphs with no large odd holes (Xingxing Yu)

Definition: Let $\mathcal{G}$ denote the class of all graphs G with the following properties:

- G is 3-connected and internally 4-connected,
- the girth of G is 5, and
- G contains no odd holes of length at least 7.

Question (Robertson 2010; Plummer and Zha 2012): Find a structural characterization of graphs in $\mathcal{G}$.

Conjecture (Plummer and Zha 2012): All graphs in $\mathcal{G}$ are 3-colorable.

17. Querying for Subgraphs (Xiaoyu He)

Suppose G is an infinite hidden Erdős-Rényi random graph $G(\mathbb{N}, p)$, $p > 0$ very small.

Let H be a fixed target graph we would like to find in G, e.g. $H = K_4$.

Problem: Let $f(H,p)$ be the number of adjacency queries needed to reveal a copy of H in G with probability at least $1/2$. What is the growth rate of $f(H,p)$ as $p \to 0^+$?

For cliques, [Conlon, Fox, Grinshpun, H. '19] proved that

- $f(K_3, p) \asymp p^{-\frac{3}{2}}$,
- $f(K_4, p) \asymp p^{-2}$,

- $f(K_5,p) \asymp p^{-\frac{8}{3}}$
- $p^{-(2-\sqrt{2})n+O(1)} \leqslant f(K_n,p) \leqslant p^{-2/3n-O(1)}$

Problem: What about K_6? Know $p^{-\frac{13}{4}} \ll f(K_6,p) \ll p^{-\frac{10}{3}}$.

For degenerate graphs [Alweiss, Ben Hamida, H., Moreira '20] proved that if H is d-degenerate ($d \geqslant 2$), then $f(H,p) = o(p^{-d})$. However, there exists a 2-degenerate H with $\frac{p^{-2}}{\log^4(1/p)} \ll f(H,p) \ll \frac{p^{-2}}{\log(1/p)}$.

Problem. For which H is $f(H,p) \asymp p^{-c}$ for some constant c?

18. Asymptotic dimension of embedded graphs (Chun-Hung Liu)

The *asymptotic dimension* of a graph class $\mathcal{F}$ is the minimum d such that there exists a function f such that $\forall G \in \mathcal{F}$ and $\forall r \in \mathbb{N}$, $V(G)$ can be colored with $d+1$ colors so that $\forall x, y \in V(G)$, if they are connected by a monochromatic path in G^r, then the distance between x, y in G is $\leqslant f(r)$.

Theorem (Gromov):

(1) The asymptotic dimension of the class of d-dimensional grids $= d$.
(2) Any infinite class of bounded degree expanders has infinite asymptotic dimension.

Theorem (Bonamy, Bousquet, Esperet, Groenland, L., Pirot, Scott):

(1) Any proper minor-closed family has asymptotic dimension $\leqslant 2$.
(2) The class of (g,k)-planar graphs has asymptotic dimension $= 2$.

Question: Does every graph class consisting of "essentially d-dimensional objects" have asymptotic dimension $\leqslant d$? $\leqslant g(d)$? $\geqslant d$? $\geqslant g(d)$?

Question: Does the class of graphs admitting book embeddings with k pages have asymptotic dimension 2?

Note added. It appears from remarks of Noga Alon and Vida Dujmović during the session that the answer to this second question is negative, as there exist bounded degree expanders with bounded page number.

19. Ramsey's theorem for matroid lines (Jim Geelen)

Conjecture. For $r \gg \ell$, if we 2-color the elements of a simple rank-r matroid M with no lines of length $> \ell$, then there is a monochromatic line.

- true for binary matroids
- true for $\mathbb{R}$-representable matroids
- true for $\ell \leqslant 3$
- natural extension to higher rank flats is also open

20. Planar graphs that are far from being 3-colorable (Louis Esperet)

For $\epsilon > 0$, a graph G is *ϵ-far* from a property $\mathcal{P}$ if one needs to delete at least $\epsilon|V(G)|$ edges from G to obtain a graph from $\mathcal{P}$.

It was proved in [A. Czumaj, M. Monemizadeh, K. Onak, C. Sohler, *Planar Graphs: Random Walks and Bipartiteness Testing*, arxiv 1407.2109] that if an n-vertex planar graph is ϵ-far from being bipartite, then G contains $\Omega(n)$ edge-disjoint odd cycles.

Question (Sohler): Is it true that if an n-vertex planar graph G is ϵ-far from being 3-colorable, then G contains $\Omega(n)$ edge-disjoint non-3-colorable subgraphs?

If true, it would imply that 3-colorability of planar graphs is testable in the sparse model, i.e., that we only need constantly many queries in this model to decide wether a planar graph is 3-colorable, or ϵ-far from being 3-colorable, with good probability.