Lecture 8 (summary)

In this lecture, we are interested in the number of degrees of freedom among subgraph densities. We prove the following theorem due to Erdős, Lovász and Spencer from 1979.

Theorem. *Let $n \in \mathbb{N}$ and let $H_1, \ldots, H_m$ be all connected graphs with at least two and at most n vertices.*

- *For every graph H with at most n vertices, there exists a function $f : [0,1]^m \to [0,1]$ such that $t(H, W) = f(t(H_1, W), \ldots, t(H_m, W))$ for every graphon W.*
- *There exists $z_0 \in [0,1]^m$ and $\varepsilon > 0$ that for every $z \in U_\varepsilon(z_0)$ there exists a graphon W such that $t(H_i, W) = z_i$ for every $i \in [m]$.*

The theorem also holds with $t(\cdot, W)$ replaced with $d(\cdot, W)$.

Observe that if H is a graph and $H_1, \ldots, H_k$ its connected components, then $t(H, W) = t(H_1, W) \times \cdots \times t(H_k, W)$; this establishes the first point (note that $t(K_1, W) = 1$).

Fix $n \geq 2$ and the graphs $H_1, \ldots, H_m$ for the rest of this summary. We introduce a family of graphons W parameterized by s_i, $i \in [m]$, and $t_{i,v}$, $i \in [m]$ and $v \in V(H_i)$. Consider s_i, $i \in [m]$, and $t_{i,v}$, $i \in [m]$ and $v \in V(H_i)$ such that

$$\sum_{i \in [m]} s_i \sum_{v \in V(H_i)} t_{i,v} \leq 1.$$

Split the unit interval $[0,1]$ to disjoint intervals $J_{i,v}$ indexed by $i \in [m]$ and $v \in V(H_i)$, set $W(x, y) = 1$ if there exist $i \in [m]$, $v \in V(H_i)$ and $w \in V(H_i)$ such that $x \in J_{i,v}$, $y \in J_{i,w}$ and $vw \in E(H_i)$. Observe that the following identity holds:

$$t(H_i, W) = \sum_{j=1}^{m} s_j^{v(H_i)} \sum_{f : V(H_i) \to V(H_j)} \prod_{vw \in E(H_i)} t_{i,f(v)} t_{j,f(w)}$$

where the product is taken over all f that are homomorphisms from H_i to H_j. In particular, $t(H_i, W)$ is a homogeneous polynomial of degree $v(H_i)$ in the s-variables and degree $|E(H_i)|$ in the t-variables. To prove the theorem, it is enough to show that there exists a choice of s_i^0, $i \in [m]$, and $t_{i,v}^0$, $i \in [m]$ and $v \in V(H_i)$ such that the Jacobian matrix of $t(H_i, W)$ viewed as functions of the s-variables is non-singular; recall that the entry in the i-th row and the j-column of the Jacobian matrix is $\frac{\partial}{\partial s_j} t(H_i, W)$.

Since each $t(H_i, W)$ is a polynomial is the s-variables and t-variables, the determinant of the Jacobian matrix is also a polynomial in these variables. In addition, the degree in the s-variables is $v(H_1) + \cdots + v(H_m) - m$ and the degree in the t-variables is the number of edges of all graphs $H_1, \ldots, H_m$. We will investigate how the terms in the determinant arise. The determinant formula involves summing over all transversals of the matrix, which we may view as a bijection $g : [m] \to [m]$. This corresponds for taking a term from $\frac{\partial}{\partial s_{g(i)}} t(H_i, W)$, which must come from the $g(i)$-th summand, and so for each homomorphism $f_{i\to g(i)}$ from H_i to $H_{g(i)}$, we obtain that the following term:

$$\prod_{i=1}^{m} s_{g(i)}^{v(H_i)-1} \prod_{v\in V(H_i)} t_{g(i),f_{i\to g(i)}(v)}.$$

We now investigate how the term

$$\prod_{i=1}^{m} s_i^{v(H_i)-1} \prod_{v\in V(H_i)} t_{i,v}$$

may arise. Since the degree of s_i is $v(H_i)-1$, the number of vertices of H_i and $H_{g(i)}$ must be the same, and since all the t-variables in the term are distinct, every $f_{i\to g(i)}$ must be injective. As the functions $f_{i\to g(i)}$ preserve edges, they must in fact be isomorphisms, which implies that $g(i)=i$. Hence, the above term can arise only by taking the diagonal entries and the coefficient at the term is the product of the numbers of isomorphisms of each of the graphs $H_1,\dots,H_m$. In particular, the determinant of Jacobian matrix is a non-zero polynomial and so there is a choice of s_i^0, $i\in[m]$, and $t_{i,v}^0$, $i\in[m]$ and $v\in V(H_i)$ for that the Jacobian matrix is non-singular.