# Lecture 4 (summary)

In this lecture, we introduce the regularity method, focusing on regularity decompositions of graphs. Szemerédi Regularity Lemma, which we will prove later, states that every graph can be decomposed into a bounded number of parts such that most them interact in a quasirandom way.

**Lemma** (Szemerédi Regularity Lemma, 1978). *For every $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $K_0 \in \mathbb{N}$ such that every graph $G$ has a vertex partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$, $k_0 \le k \le K_0$, such that*

- *$|V_0| \le \varepsilon |V(G)|$ and $|V_1| = \cdots = |V_k|$, and*
- *all pairs of parts $V_i$ and $V_j$, $1 \le i < j \le k$, except for at most $\varepsilon k^2$ pairs satisfy that*

$$\left| \frac{e(A,B)}{|A|\,|B|} - \frac{e(V_i,V_j)}{|V_i|\,|V_J|} \right| \le \varepsilon$$

  *holds for all subsets $A \subseteq V_i$ and $B \subseteq V_j$ with $|A| \ge \varepsilon |V_i|$ and $|B| \ge \varepsilon |V_j|$,*

*where $e(X,Y)$ denotes the number of edges between sets $X$ and $Y$.*

The pairs $V_i$ and $V_j$ that satisfy the property given in the second bullet point of the lemma are referred to as *$\varepsilon$-regular.*

To illustrate the statement of Szemerédi Regularity Lemma, we prove the Graph Removal Lemma for triangles. In its full generality, the Graph Removal Lemma reads as follows.

**Lemma** (the Graph Removal Lemma). *For every $\varepsilon > 0$ and every graph $H$, there exists $\delta > 0$ such that every $n$-vertex graph $G$ satisfies (at least) one of the following:*

- *$G$ contains at least $\delta n^{|V(H)|}$ copies of $H$, or*
- *there exists a set $F \subseteq E(G)$ such that $|F| \le \varepsilon n^2$ and $G \setminus F$ is $H$-free.*

We proved the Graph Removal Lemma when $H = K_3$ in the lecture and we include a proof sketch here.

*Sketch of proof for $H = K_3$.* Fix $\varepsilon \in (0,1)$, and apply Szemerédi Regularity Lemma with $\varepsilon_R = \varepsilon/100$ and $k_0 = \lceil \varepsilon_R^{-1} \rceil$ to get $K_0$. Let $G$ be an $n$-vertex graph, and apply Szemerédi Regularity Lemma to get a partition of $V(G)$ to $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ as in the statement of Szemerédi Regularity Lemma. Let $d_0 = \varepsilon/10$ and construct an auxiliary graph $R$ with vertices corresponding to the parts $V_1, \ldots, V_k$; a pair vertices corresponding to $V_i$ and $V_j$ is joined by an edge if $V_i$ and $V_j$ is an $\varepsilon$-regular pair and $e(V_i, V_j) \ge d_0 |V_i|\,|V_j|$.

If $R$ has no triangle, then set $F$ to contain all edges incident with a vertex of $V_0$, all edges inside the parts $V_1, \ldots, V_k$, all edges between any pair of parts $V_i$ and $V_j$ such that

$V_i$ and $V_j$ is not an $\varepsilon$-regular pair or $e(V_i, V_j) < d_0|V_i|\,|V_j|$. Note that the graph $G \setminus F$ has no triangle. A simple counting argument shows that $|F| \le \varepsilon n^2$.

If $R$ has a triangle, we may assume by changing the indices of the parts that every pair of the parts $V_1$, $V_2$ and $V_3$ is an $\varepsilon$-regular pair and $e(V_i, V_j) \ge d_0|V_i|\,|V_j|$ for all $1 \le i < j \le 3$. Using that $V_1$ and $V_i$ for $i \in \{2, 3\}$, we show that all but $\varepsilon|V_1|$ vertices of $V_1$ have at least

$$\left(\frac{e(V_1, V_i)}{|V_1|\,|V_i|} - \varepsilon\right)|V_i| \ge \frac{d_0}{2}\,|V_i|$$

neighbors in $V_i$. Consider a vertex $w$ of $V_1$ with at least $\frac{d_0}{2}\,|V_2|$ neighbors in $V_2$ and at least $\frac{d_0}{2}\,|V_3|$ neighbors in $V_3$, and let $A$ and $B$ be the neighbors of $w$ in $V_2$ and $V_3$, respectively. Since $|A| \ge \varepsilon_R|V_2|$ and $|B| \ge \varepsilon_R|V_3|$, we obtain that $e(A, B) \ge (d_0 - \varepsilon)|A|\,|B|$, i.e., $w$ is in at least $e(A, B)$ triangles. Since the size of each set $V_1$, $V_2$ and $V_3$ is at least $(1 - \varepsilon_R)n/K_0$, it follows that the statement of the lemma holds with $\delta = \frac{d_0^3}{32K_0^3}$. $\square$

Proving the Graph Removal Lemma in full generality as an exercise, which is split into two steps (each including a different technical challenge to overcome).

**Exercise.** *Prove the Graph Removal Lemma when $H$ is any complete graph.*

**Exercise.** *Prove the Graph Removal Lemma for all graphs $H$.*