Lecture 2 (summary)

In this lecture, we first finish the proof on the existence of a limit permuton, which we started in Lecture 1. In particular, we continue using the notation introduced in Lecture 1. Fix a permutation σ and consider k such that $2^k > |\sigma|$. Consider the event that no two points among $|\sigma|$ points $(x_1, y_1), \ldots, (x_{|\sigma|}, y_{|\sigma|})$ chosen according to the measure μ, which we constructed in Lecture 1, have their x-coordinates in the same dyadic interval of length 2^{-k}, i.e., $\lfloor 2^k x_i \rfloor \neq \lfloor 2^k x_j \rfloor$ for all $1 \leq i < j \leq |\sigma|$, or their y-coordinates in the same dyadic interval of length 2^{-k}, i.e., $\lfloor 2^k y_i \rfloor \neq \lfloor 2^k y_j \rfloor$ for all $1 \leq i < j \leq |\sigma|$; let ρ_k be the probability of this event. Observe that

$$\rho_k \leq 2\binom{|\sigma|}{2}\frac{1}{2^k}.$$

Next observe that the probability that a μ-random permutation is σ and the above introduced event does not occur is equal to

$$\sum_{1 \leq x_1 < \cdots < x_{|\sigma|} \leq 2^k} \sum_{1 \leq y_1 < \cdots < y_{|\sigma|} \leq 2^k} |\sigma|! \prod_{i=1}^{|\sigma|} A^k_{x_i, y_{\sigma(i)}}.$$

It follows that

$$\left| d(\sigma, \mu) - \sum_{1 \leq x_1 < \cdots < x_{|\sigma|} \leq 2^k} \sum_{1 \leq y_1 < \cdots < y_{|\sigma|} \leq 2^k} |\sigma|! \prod_{i=1}^{|\sigma|} A^k_{x_i, y_{\sigma(i)}} \right| \leq \rho_k.$$

Similarly, if π_n is a permutation in the convergent sequence of permutations with size divisible by 2^k, the probability that a randomly sampled subpermutation of size $|\sigma|$ have no two indices or images within the same dyadic interval of length $2^{-k}n$ is ρ_k and so we obtain that

$$\left| d(\sigma, \pi_n) - \sum_{1 \leq x_1 < \cdots < x_{|\sigma|} \leq 2^k} \sum_{1 \leq y_1 < \cdots < y_{|\sigma|} \leq 2^k} |\sigma|! \prod_{i=1}^{|\sigma|} \left(A^k_n\right)_{x_i, y_{\sigma(i)}} \right| \leq \rho_k,$$

which implies that

$$\left| \lim_{n \to \infty} d(\sigma, \pi_n) - \sum_{1 \leq x_1 < \cdots < x_{|\sigma|} \leq 2^k} \sum_{1 \leq y_1 < \cdots < y_{|\sigma|} \leq 2^k} |\sigma|! \prod_{i=1}^{|\sigma|} A^k_{x_i, y_{\sigma(i)}} \right| \leq \rho_k.$$

We now combine the estimates obtained above to conclude that

$$\left| d(\sigma, \mu) - \lim_{n \to \infty} d(\sigma, \pi_n) \right| \leq 2\rho_k.$$

Since the $\rho_k \to 0$, we obtain that μ is a limit permuton of the sequence $(\pi_n)_{n \in \mathbb{N}}$.

As a preparation for the next lecture, we relate a certain integral to densities of permutations in a permuton. Fix a permuton μ and define $F(x,y) = \mu([0,x] \times [0,y])$. We wish to study the integral

$$\int_{[0,1]^2} F(x,y)\,\mathrm{d}\lambda,$$

where λ is the uniform measure. Note that the value of the integral is the probability that $x \le x'$ and $y \le y'$ for $(x,y) \sim \mu$ and $(x',y') \sim \lambda$, which is the probability that $x \le x'$ and $y \le y'$ for $(x,y) \sim \mu$, $(x',y'') \sim \mu$ and $(x'',y') \sim \mu$ (as the measure μ has uniform marginals). Since the latter probability depends only the mutual position of the three points (x,y), (x',y'') and (x'',y'), which is fully captured by a permutation the three points yield, we obtain that the value of the integral is equal to

$$\frac{1}{3}d(123,\mu) + \frac{1}{3}d(132,\mu) + \frac{1}{3}d(213,\mu) + \frac{1}{6}d(231,\mu) + \frac{1}{6}d(312,\mu) + \frac{1}{6}d(321,\mu).$$