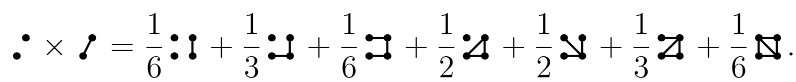
Lecture 10 (summary)

In this lecture, we introduce basic formalism of the flag algebra method in the setting of graphs. While we will use graph limits to define the formalism, it should be emphasized that the flag algebra method is independent of a chosen analytic representation of a convergent sequence.

Let $\mathcal{A}$ be the algebra of formal linear combinations of graphs with real coefficients with the natural operations of addition and multiplication by a scalar. For a graphon W, we define $t_W : \mathcal{A} \to \mathbb{R}$ as $t_W(G) = d(G, W)$ for each single graph G and extend the definition to $\mathcal{A}$ linearly. It is clear that $t_W(a + a') = t_W(a) + t_W(a'$ and $t_W(\alpha a) = \alpha t_W(a)$ for any elements $a, a' \in \mathcal{A}$ and any real $\alpha \in \mathbb{R}$. We next define multiplication of elements from $\mathcal{A}$ in a way that commutes with t_W (for every graphon W). For two graphs G and G', set

$$G \times G' = \sum_{H, v(H)=v(G)+v(G')} \frac{\left|\left\{A \in \binom{V(H)}{v(G)},\ H[A] = G \text{ and } H[V(H) \setminus A] = G'\right\}\right|}{\binom{v(G)+v(G')}{v(G)}} H$$

and extend the definition $\times$ to $\mathcal{A}$ linearly. For example, it holds that



We obtain that $t_W(a \times a') = t_W(a) t_W(a')$ for any two elements $a, a' \in \mathcal{A}$.

We will use the shorthand notation $a = a'$ to represent $t_W(a) = t_W(a')$, for all graphons W and $a \geq a'$ to represent $t_W(a) \geq t_W(a')$ for all graphons W (where $a, a' \in \mathcal{A}$). Similarly, for $a \in \mathcal{A}$ and $\alpha \in \mathbb{R}$, we write $a = \alpha$ and $a \geq \alpha$ if $t_W(a) = \alpha$ and $t_W(a) \geq \alpha$ for all graphons W. It is easy to show that if G is a graph and $k \geq v(G)$, then

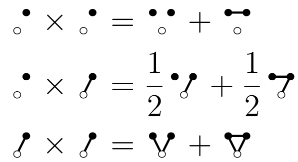
$$G - \sum_{H, v(H)=k} d(G, H) H = 0.$$

Let $\mathcal{A}^\bullet$ to be the algebra of formal linear combinations of rooted graphs, i.e., graphs with a single distinguished vertex referred to as the root. For a graphon W, we define a probabilistic distribution on maps $t^\bullet_W : \mathcal{A}^\bullet \to \mathbb{R}$ as follows: pick $x_0 \in [0, 1]$ uniformly at random, define

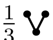
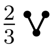
$$t^{x_0}_W = \frac{k!}{|\mathrm{Aut}^\bullet(H)|} \int_{[0,1]^k} \prod_{v_i v_j \in E(H)} W(x_i, x_j) \prod_{v_i v_j \notin E(H)} (1 - W(x_i, x_j))\, \mathrm{d}x_{[k]}$$

for every rooted graph H (where $\mathrm{Aut}^\bullet(H)$ is the group of automorphisms of H fixing the root), and extend $t^{x_0}_W$ to $\mathcal{A}^\bullet$ linearly. We would like to emphasize that while $t^\bullet_W(a)$ is a random variable, it is good to keep in mind that the choice of the root determine the values of $t^\bullet_W$ for all elements $a \in \mathcal{A}$ at once. Similarly as in the unrooted case, we define $G \times G'$ for two rooted graphs G and G' as the linear combination of rooted graphs H

with $v(G) + v(G') - 1$ vertices with coefficients equal to the number of partitions of the $v(G) + v(G') - 2$ non-root vertices of H to the parts with $v(G) - 1$ and $v(G') - 1$ vertices inducing G and G' divided by $\binom{v(G)+v(G')-2}{v(G)-1}$. We extend the definition of the product from pairs of rooted graphs to all elements of $\mathcal{A}^\bullet$ linearly and observe that $t^\bullet_W(a \times a') = t^\bullet_W(a)t^\bullet_W(a')$ for any $a, a' \in \mathcal{A}^\bullet$. In particular, it holds that

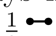
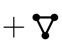
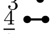
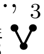


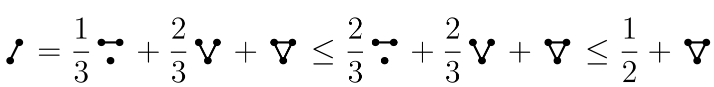
As in the unrooted case, every k-vertex rooted graph can be expressed as a linear combination of ℓ-vertex rooted graphs for every $\ell \geq k$.

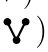
Our next goal is to relate the expected value of $t^\bullet_W(a)$ to values of t_W. Specifically, we seek to define an operation $[\![a]\!]$ such that $\mathbb{E}_x t^x_W(a) = t_W([\![a]\!])$ for every graphon W. Following the definition of t^x_W and $\mathbb{E}_x t^x_W(a)$, we deduce that $[\![H]\!]$ for a rooted graph H should be $\alpha H'$ where H' is the graph H with no vertex distinguished as the root and α is the the probability that H' becomes H when rooting at a random vertex, and we then extend $[\![\cdot]\!]$ linearly to all elements of $\mathcal{A}^\bullet$. For example, $[\![$  $]\!] = \frac{1}{3}$  and $[\![$  $]\!] = \frac{2}{3}$ .

As an example of the use of the just introduced formalism, we deduce the following asymptotic version of Mantel's Theorem.

Theorem. *If W is a graphon with $d(K_3, W) = 0$, then $d(K_2, W) \leq 1/2$.*

Observe that $0 \leq [\![($  $-$ $)^2]\!]$ as the right side of the inequality is the expected value of a square and so always non-negative. Expanding the right side, we obtain that $0 \leq$ $+$  $- \frac{1}{3}$ $- \frac{1}{3}$ , i.e., $\frac{1}{3}$  $+ \frac{1}{3}$  $-$  $-$  ≤ 0. Since it holds that  $+$  $+$  $+$  $= 1$, it follows that $\frac{4}{3}$  $+ \frac{4}{3}$  ≤ 1. We now proceed as follows:



Hence, if $t_W(K_3) = d(K_3, W) = 0$, then $t(K_2, W) = d(K_2, W) \leq 1/2$ as desired. Moreover, if $t(K_2, W) = 1/2$ (when $t_W(K_3) = 0$), then all inequalities are equalities and we obtain that $t_W($  $) = 0$ and $t^x_W($  $-$  $) = 0$ for almost all $x \in [0, 1]$, i.e., $t^x_W($  $) = 1/2$ for almost all $x \in [0, 1]$. We leave as an exercise to deduce the structure of graphons that attain the equality.

Exercise. *Show that if W is a graphon with $d(K_2, W) = 1/2$ and $d(K_3, W) = 0$, then there exists a partition of $[0, 1]$ to two disjoint measurable sets A and B such that $W(x, y) = 0$ for almost every $(x, y) \in A^2 \cup B^2$ and $W(x, y) = 1$ for almost every $(x, y) \in [0, 1]^2 \setminus (A^2 \cup B^2)$.*

The asymptotic version of Mantel's Theorem shown earlier in this lecture implies that every n-vertex K_3-free graph has at most $\frac{1}{2}\binom{n}{2} + o(n^2)$ edges. The following statement is left an exercise of another simple use of the flag algebra method.

Exercise. *Show that any 2-edge-coloring of K_n contains at least $\frac{1}{4}\binom{n}{3} + o(n^3)$ monochromatic triangles.*