Lecture 1 (summary)

In this lecture, we start building limit theory of permutations. A *permutation of size* k is a bijective mapping from $[k]$ to $[k]$, where $[k]$ denotes the set of the first k positive integers. If π is a permutation, then $|\pi|$ denotes its size. Permutations will be understood from the combinatorial point of view only. In particular, a permutation of size k can be thought as two linear orders on the set $[k]$: the first is the usual order on $[k]$ and the other is given by the images of the elements. A *subpermutation* of a permutation π induced by $A =\subseteq [k]$ is a permutation σ of order $|A|$ such that $\sigma(i) < \sigma(j)$ iff $\pi(a_i) < \pi(a_j)$ where $a_1 < \cdots < a_{|A|}$ are the elements of A. We remark that subpermutations are usually called patterns, however, the terminology used is closer to that of graphs to be seen later in the course.

The *density* of a permutation σ in a permutation π, which is denoted by $d(\sigma, \pi)$, is the probability that the subpermutation of π induced by a randomly uniform $|\sigma|$-element subset of π is σ. We set $d(\sigma, \pi) = 0$ when $|\sigma| > |\pi|$. We say that a sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations is *convergent* if $|\pi_n| \to \infty$ and $d(\sigma, \pi_n)$ converges for every permutation σ. Observe that the following holds for all permutations σ and π and every integer k such that $|\sigma| \le k \le |\pi|$:

$$d(\sigma, \pi) = \sum_{\sigma' \in S_k} d(\sigma, \sigma') d(\sigma', \pi).$$

Once we have built the necessary framework, the first result that we will prove using permutation limits will be the following theorem.

Theorem. *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of permutations such that $|\pi_n| \to \infty$. If it holds that $\lim_{n\to\infty} d(\sigma, \pi_n) = 1/24$ for every permutation σ of size 4, then the sequence $(\pi_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} d(\sigma, \pi_n) = 1/|\sigma|!$ for every permutation σ.*

The above statement is an analogue of the C_4 statement we saw in Lecture 0 on quasirandom graphs and was stated as an open problem (with 4 replaced with any constant) by Graham around 2000. The statement was proven using the theory of permutation limits about a decade later, however, an equivalent statement was proven in statistics in relation to non-parametric test of independence by Hoeffding already in 1948 (with 4 replaced by 5).

Convergent sequences of permutations can be associated with the following analytic limit object: a *permuton* is a probability measure μ on $[0, 1]^2$ with uniform marginals, i.e., $\mu([a, b] \times [0, 1]) = \mu([0, 1] \times [a, b]) = b - a$. The original limit object developed by Hoppen, Kohayakawa, Moreira, Ráth and Sampaio was more complex although mathematically equivalent, and it became standard to use permutons only to represent permutation limits. If μ is a permuton, then a *μ-random permutation* of size k is obtained as follows. First, sample k points $(x_1, y_1), \dots, (x_k, y_k)$ according to μ; note that the k points have mutually distinct x-coordinates or y-coordinates with probability one. Assume (by renaming the points) that $x_1 < \cdots < x_k$ and define the permutation π of size k such that $\pi(i) < \pi(j)$ iff $y_i < y_j$. Finally, if σ is a permutation of size k, then the *density* of σ in μ, which is denoted by $d(\sigma, \mu)$, is the probability that a μ-random permutation of size k is σ.

We say that a permuton μ is a limit of a convergent sequences $(\pi_n)_{n\in\mathbb{N}}$ of permutations if $d(\sigma,\mu) = \lim_{n\to\infty} d(\sigma,\pi_n)$ for every permutation σ. In the rest of the lecture, we prepare for proving the following theorem.

Theorem. *Every convergent sequence of permutations has a limit permuton.*

We leave the the following as an exercise.

Exercise. *The limit permuton of every convergent sequence of permutations is unique.*

We first establish the following lemma.

Lemma. *Let $(\pi_n)_{n\in\mathbb{N}}$ be a convergent sequence of permutations. There exists a convergent sequence $(\pi'_n)_{n\in\mathbb{N}}$ of permutations such that for every k, there exists n_k that the size of every π'_n for $n \ge n_k$ is divisible by 2^k and $\lim_{n\to\infty} d(\sigma,\pi'_n) = \lim_{n\to\infty} d(\sigma,\pi_n)$.*

Note that it is enough to prove the existence of a limit permuton for convergent sequence of permutations having the property stated in the lemma above. Fix such a convergent sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations and let n_k be such that the size of every π_n for $n \ge n_k$ is divisible by 2^k. Define matrices $A^k_n \in [0,1]^{2^k\times 2^k}$ for $k \in \mathbb{N}$ and $n \ge n_k$ as

$$(A^k_n)_{i,j} = \frac{\left|\left\{x \text{ s.t. } \frac{(i-1)|\pi_n|}{2^k} < x \le \frac{i|\pi_n|}{2^k} \text{ and } \frac{(j-1)|\pi_n|}{2^k} < \pi_n(x) \le \frac{j|\pi_n|}{2^k}\right\}\right|}{|\pi_n|}.$$

It can be shown that the matrices A^k_n coordinate-wise converge for every $k \in \mathbb{N}$. However, since the proof of this statement is technical, we rather consider a subsequence of $(\pi_n)_{n\in\mathbb{N}}$ such that the matrices A^k_n coordinate-wise converge for every $k \in \mathbb{N}$; let A^k be the limit matrix.

We say that $X \subseteq [0,1)^2$ is *dyadic* of order $k \in \mathbb{N}$ if X is of the form $\left[\frac{i-1}{2^k}, \frac{i}{2^k}\right) \times \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right)$ for some $i, j \in [2^k]$. Consider the family $\mathcal{A}$ of subsets of $[0,1)^2$ that are finite unions of dyadic. Note that $\mathcal{A}$ is closed under taking complements, finite unions and finite intersections, i.e., $\mathcal{A}$ is an algebra of sets. Observe that every set X in $\mathcal{A}$ can be expressed as a union of dyadic sets of the same order k and define a premeasure μ_0 by setting $\mu_0(X)$ to be the sum of the corresponding coordinates of the matrix A^k. Since it holds that

$$\mu_0\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i)$$

for every sequence $(A_i)_{i\in\mathbb{N}}$ of disjoint sets from $\mathcal{A}$ such that $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$, Carathéodory's extension theorem implies that there exists a measure μ on the σ-algebra of Borel subsets of $[0,1)^2$ (and so on the σ-algebra of Borel subsets of $[0,1]^2$) such that

$$\mu\left(\left[\frac{i-1}{2^k}, \frac{i}{2^k}\right) \times \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right)\right) = A^k_{ij}$$

for every $k \in \mathbb{N}$ and all $i, j \in [2^k]$. In the next lecture, we show that this measure μ is the limit permuton of the sequence $(\pi_n)_{n\in\mathbb{N}}$.