Lecture 0 (summary)

This lecture is focused on briefly recalling some classical results in combinatorics, which will be seen through lenses of the theory of combinatorial limits in the course. Combinatorial objects that we particularly focus on will be graphs, permutations and hypergraphs.

Many problems that will be looked at will come from extremal combinatorics. Let us recall the following two classical results in extremal graph theory.

Theorem (Mantel's Theorem, 1907). *The maximum number of edges in a K_3-free n-vertex graph is $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.*

Theorem (Turán's Theorem, 1941). *The maximum number of edges in a K_{k+1}-free n-vertex graph is $t(n,k)$ where $t(n,k)$ is the number of edges of the complete k-partite n-vertex graph with parts of sizes $\lfloor \frac{n}{k} \rfloor$ and $\lceil \frac{n}{k} \rceil$.*

The flag algebra method, which will be presented in the course, yields a computer assisted way of establishing asymptotically true inequalities. For example, we will see how the method yields the following asymptotic version of Mantel's Theorem: the number of edges in a K_3-free n-vertex graph is at most $\frac{1}{2}\binom{n}{2} + o(n^2)$.

If G and H are graphs, then a function f from $V(H)$ to $V(G)$ is *homomorphism* if $f(u)f(v) \in E(G)$ for every $uv \in E(H)$. We write $t(H,G)$ for the probability that a uniformly random mapping from $V(H)$ to $V(G)$ is a homomorphism. Recall that the Erdős-Rényi random graph $G_{n,p}$ is the n-vertex graph where any pair of vertices is joined by an edge with probability p independently of the other pairs. Note that the expected value of $t(H, G_{n,p})$ is $p^{|E(H)|}$ (if $n \geq |V(H)|$). Graphs G with $t(H,G)$ close to $p^{|E(H)|}$ (the expected value in the Erdős-Rényi random graph) are called quasirandom. Classical results on quasirandom graphs by Rödl, Thomason and culminating with the work by Chung, Graham and Wilson yield that the following statements are equivalent for every sequence $(G_n)_{n\in\mathbb{N}}$ for graphs with $V(G_n) \to \infty$:

- $\lim\limits_{n\to\infty} t(H, G_n) = p^{|E(H)|}$ for every graph H,
- $\lim\limits_{n\to\infty} t(K_2, G_n) = p$ and $\lim\limits_{n\to\infty} t(C_4, G_n) = p^4$,
- for all $\varepsilon > 0, \delta > 0$, there exists n_0 such that

  $$\left| \frac{\left|E(G) \cap \binom{X}{2}\right|}{\binom{|X|}{2}} - p \right| \leq \delta$$

  for every $n \geq n_0$ and every $X \subseteq V(G_n)$ with $|X| \geq \varepsilon |V(G_n)|$,
- for all $\varepsilon > 0$, there exists n_0 such that the eigenvalues $\lambda_1, \ldots, \lambda_{|V(G_n)|}$ of the adjacency matrix of G_n, $n \geq n_0$, satisfy that $|\lambda_1 - p|V(G_n)|| \leq \varepsilon |V(G_n)|$ and $|\lambda_i| \leq \varepsilon |V(G_n)|$ for $i \geq 2$.

We will prove the above equivalences using the theory of graph limits.

When building the theory of graph limits, we will use Szemerédi's Regularity Lemma, one of the cornerstones of modern graph theory. From the Graph Removal Lemma, the "counting" corollary of Szemerédi's Regularity Lemma, we will derive Roth's Theorem, one of the fundamental results in combinatorial number theory:

Theorem (Roth's Theorem, 1953)**.** *For every $\varepsilon > 0$, there exists n_0 such that every subset $A \subseteq \{1, \ldots, n\}$, $n \geq n_0$, with at least εn elements contains a 3-term arithmetic progression.*