Polynomial Identity Testing and Circuit Lower Bounds

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based on papers by

Nisan & Wigderson, 1994
Kabanets & Impagliazzo, 2003

Randomised algorithms

- For some problems (polynomial identity testing) we know an efficient randomised algorithm, but not a deterministic one.
- However nobody proved $P \subsetneq BPP$ yet.
  It is possible that $P = BPP$.
- There is a connection between *hardness* and *randomness*:
  if we have a hard function, we can use it to *derandomize* BPP.
- Until recently, it was not known whether the converse holds.
  Kabanets & Impagliazzo showed that it does.
- This is bad, since non-trivial circuit lower-bounds are a long-standing open problem.

Pseudo-random generators

$G : \{0,1\}^{\ell(n)} \to \{0,1\}^n$ is a *pseudo-random generator* iff for any circuit C of size n:

$$|P[C(r) = 1] - P[C(G(x)) = 1]| < \frac{1}{n},$$

where x, r are chosen uniformly.

Having a pseudo-random generator, we can *derandomize* BPP:

- instead of n random bits, plug a pseudo-random sequence (acceptance prob. changed only slightly)
- check all $2^{\ell(n)}$ random seeds

Hard functions

$f_n : \{0,1\}^n \to \{0,1\}$ has *hardness h*
iff for any circuit C of size h:

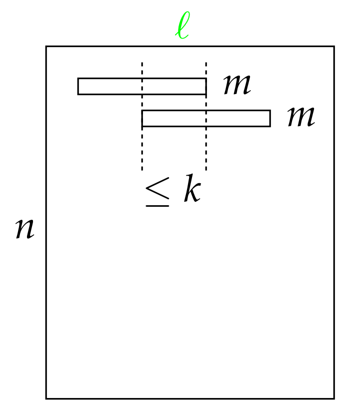
$$\left| P[C(x) = f(x)] - \frac{1}{2} \right| < \frac{1}{2h},$$

where x is chosen uniformly.

Hard functions can be used to build pseudo-random generators:

- take $\ell(n)$ truly random bits
- evaluate f on n subsets of them
- if these subsets have small intersection, then the results are hardly correlated

Nearly disjoint sets



System of sets $\{S_1, \ldots, S_n\}$, where $S_i \subset \{1, \ldots, \ell\}$ is a *(k, m)-design* if:

- $|S_i| = m$
- $|S_i \cap S_j| \leq k$

For every $m \in \{\log n, \ldots, n\}$, there exists an $n \times \ell$ matrix which is a $(\log n, m)$-design, where $\ell = O(m^2)$.
(If $m = O(\log n)$, then even $\ell = O(m)$ is enough.)

Assume m is a prime power. Take $S_q = \{\langle x, q(x)\rangle \,|\, x \in GF(m)\}$, where q has degree at most $\log n$. Can be computed in log-space.

Nisan & Wigderson, 1994

Let f have *hardness* $\geq n^2$ and S be a $(\log n, m)$-*design*. Then $G : \{0,1\}^\ell \to \{0,1\}^n$ given by $G(x) = f_S(x)$ is a pseudo-random generator.

1. Assume a circuit C distinguishes random r and $y = G(x)$ w.p. $> \frac{1}{n}$.
Let $p_i = P[C(z) = 1]$, where $z = y_1 \ldots y_i r_{i+1} \ldots r_n$.
There must be i such that $p_{i-1} - p_i > \frac{1}{n^2}$.

2. Build a circuit D that predicts y_i from $y_1 \ldots y_{i-1}$ w.p. $\geq \frac{1}{2} + \frac{1}{n^2}$
D evaluates $C(y_1 \ldots y_{i-1}, r_i \ldots r_n)$ and returns r_i iff $C = 1$.

3. Assume w.l.o.g. $S_i = \{1, \ldots, m\}$, then $y_i = f(x_1 \ldots x_m)$.
Since y_i does not depend on other bits, there exists some *assignment of* $x_{m+1} \ldots x_\ell$ preserving the prediction prob.

4. After *fixing*, every $y_1 \ldots y_{i-1}$ depends only on $\log n$ variables, hence can be computed from x as a CNF of size $O(n)$.

5. Plug computed $y_1 \ldots y_{i-1}$ into D and obtain a circuit predicting y_i from x w.p. $\geq \frac{1}{2} + \frac{1}{n^2}$.

This contradicts that f has hardness $\geq n^2$.

Hardness-randomness tradeoff

If there exists a function computable in $E = \mathrm{DTIME}(2^{\mathrm{O}(n)})$ that cannot be approximated by

1. polynomial-size circuits, then
   BPP $\subset \bigcap_{\varepsilon>0} \mathrm{DTIME}(2^{n^{\varepsilon}})$.

2. circuits of size $2^{n^{\varepsilon}}$ for some $\varepsilon > 0$, then
   BPP $\subset \mathrm{DTIME}(2^{(\log n)^{c}})$ for some constant c.

3. circuits of size $2^{\varepsilon n}$ for some $\varepsilon > 0$, then
   BPP $=$ P.
   (We need to use $(\log n, m)$-design with $\ell = \mathrm{O}(m)$.)

Impagliazzo & Wigderson, 1997

If some function in E has circuit complexity $2^{\Omega(n)}$, then BPP $=$ P.

- Similar claim as NW.3, but assuming hardness in the *worst-case*. NW needed hardness on the *average*.
- Convert mildly hard function f to almost unpredictable function. Yao's XOR-Lemma: $f(x_1) \oplus \cdots \oplus f(x_k)$ is hard to predict, when x_i are independent.
- Use expanders to reduce the need for random bits.

Is circuit lower bound needed?

- f is in BPP, if there is a randomised algorithm with error $\leq \frac{1}{3}$ on *every input*
- f is in promise-BPP, if there is a randomised algorithm with error $\leq \frac{1}{3}$ on *some subset of inputs*, and we do not care the acceptance prob. on other inputs

[Impagliazzo & Kabanets & Wigderson, 2002]

Promise-BPP $=$ P implies NEXP $\not\subset$ P/poly (circuit lower bound!).

[Kabanets & Impagliazzo, 2003]

BPP $=$ P implies super-polynomial *arithmetical* circuit lower bound for NEXP.

Prerequisites of [KI03]

- [Valiant, 1979] Perm is #P-complete
  - $\text{Perm}(A) = \sum_\sigma \prod_{i=1}^n a_{i,\sigma(i)}$
  - #P is a class counting the number of solutions
- [Toda, 1991] $\mathrm{PH} \subset \mathrm{P}^{\#\mathrm{P}}$
- [Impagliazzo & Kabanets & Wigderson, 2002]
  NEXP $\subset$ P/poly $\Longrightarrow$ NEXP $=$ MA

If NEXP $\subset$ P/poly, then

1. NEXP $=$ MA $\subset$ PH $\subset \mathrm{P}^{\#\mathrm{P}} \Longrightarrow$ Perm is NEXP-hard

2. Perm $\in$ EXP $\subset$ NEXP $\Longrightarrow$ Perm is NEXP-complete

Polynomial identity testing

- is testing whether a given polynomial is identically zero
- is in co-RP: take a random point and evaluate the polynomial. If the field is big enough, we get nonzero with high prob.

Can test whether a given *arithmetical circuit* p_n computes Perm:
Input: p_n on $n \times n$ variables, let p_i be its restriction to $i \times i$ variables.

- test $p_1(x) = x$ (by the method above)
- for $i \in \{2, \ldots, n\}$, test $p_i(X) = \sum_{j=1}^{i} x_{1,j} p_{i-1}(X_j)$, where X_j is the j-th minor

If all tests pass, then $p_n =$ Perm.

Circuit lower bounds from derandomization

- Suppose that *polynomial identity testing is in P*.
- If *Perm is computable by polynomial-size arithmetic circuits*, then Perm $\in$ NP:
  1. guess the circuit for Perm
  2. verify its validity
  3. compute the result
- If *NEXP* $\subset$ *P/poly*, then Perm is NEXP-complete.

Contradiction with nondeterministic time hierarchy theorem!

Main result of KI03

If BPP $=$ P, or even BPP $\subset$ NSUBEXP $= \bigcap_{\varepsilon>0} \mathrm{NTIME}(2^{n^{\varepsilon}})$, then

1. Perm does not have polynomial-size arithmetical circuits, **or**

2. NEXP $\not\subset$ P/poly

Summary

- [NW94] Average circuit lower bounds imply derandomization
- [IW97] Worst-case circuit lower bounds imply derandomization
- [KI03] Derandomization implies circuit lower bounds