Introduction to big Ramsey degrees

Part 1: Big Ramsey degrees of rationals and Rado graph

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Prague

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Tutorial overview

I would like to cover some old&new results in the area of big Ramsey degrees:

1. Big Ramsey degrees of rationals and Rado graph.
2. Recent progress in the area
    1. Big Ramsey degrees of Triangle-free graphs
    2. New Ramsey theorem for trees with successor operation.
3. Applications of the new Ramsey theorem
    1. Easy proof of unrestricted Nešetřil–Rödl or Abramson–Harington theorem
    2. Big Ramsey degrees of structures forbidding bigger substructures.

Ramsey theorem

Theorem (Infinite Ramsey Theorem, 1930)

$$\forall_{p,k \geq 1} : \omega \longrightarrow (\omega)^{p}_{k,1}.$$

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Definition (Erdős–Rado partition arrow)

$N \longrightarrow (n)^p_{k,t}$ means:
For every partition of $\binom{N}{p}$ into k classes (colours) there exists $X \in \binom{N}{n}$ such that $\binom{X}{p}$ belongs to at most t parts.

($t = 1$ means that $\binom{X}{p}$ is monochromatic.)

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In 1970's a concept of structural Ramsey theory was introduced. A Ramsey theorem can be seen as a theorem about the class of linear orders.

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Let $\mathcal{O}$ be the class of all finite linear orders.

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of structure **A** to structure **B**.

Definition (Leeb's generalization of the Erdős–Rado partition arrow)

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$ means:

For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

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Sierpiński: not true for $|O| = 2$.

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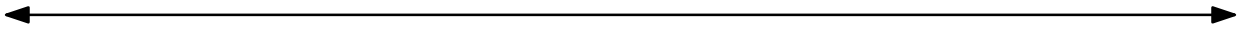

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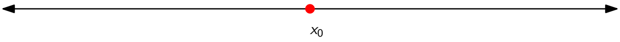

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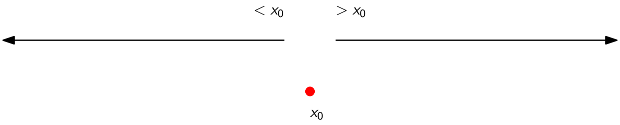

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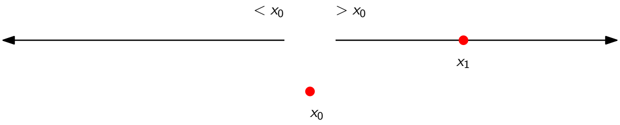

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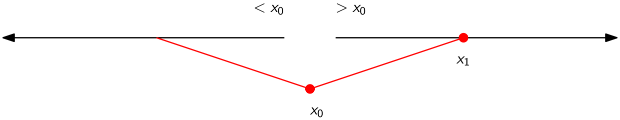

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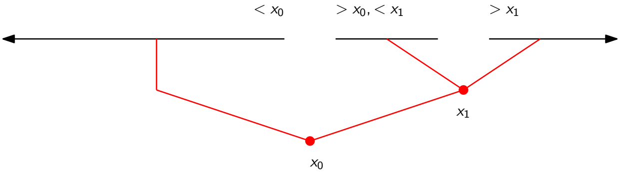

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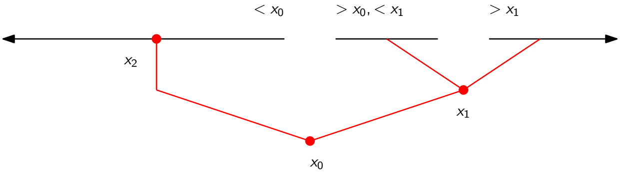

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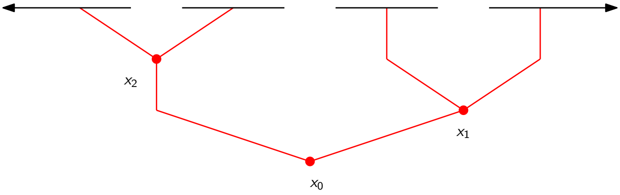

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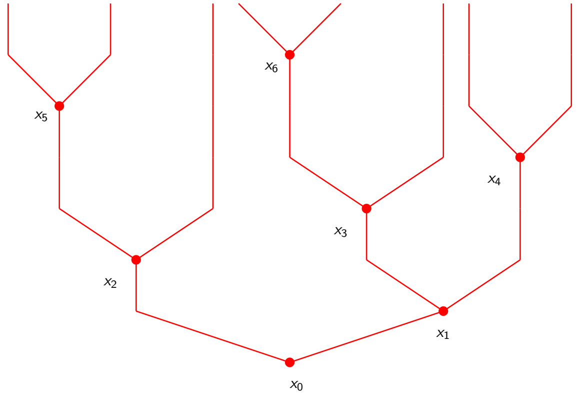

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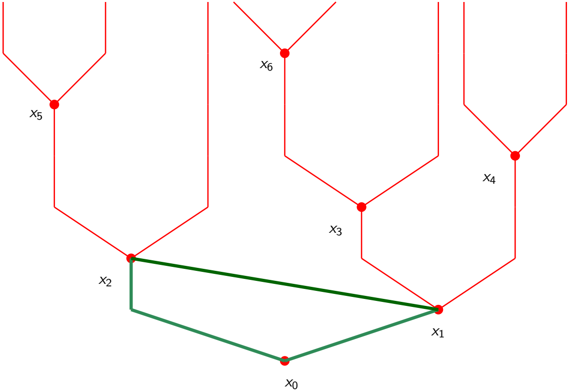

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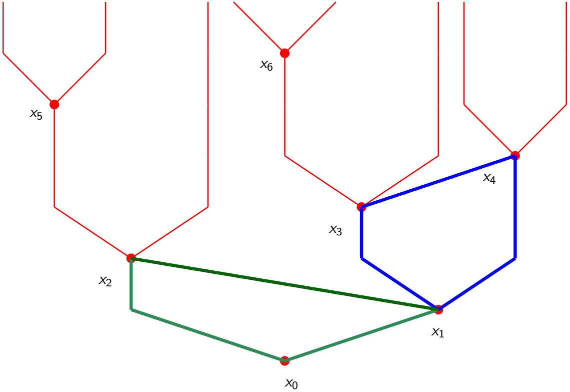

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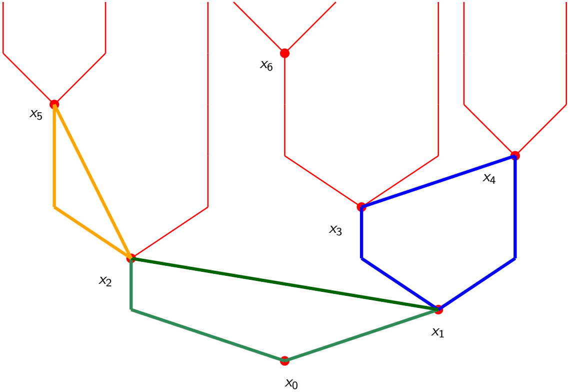

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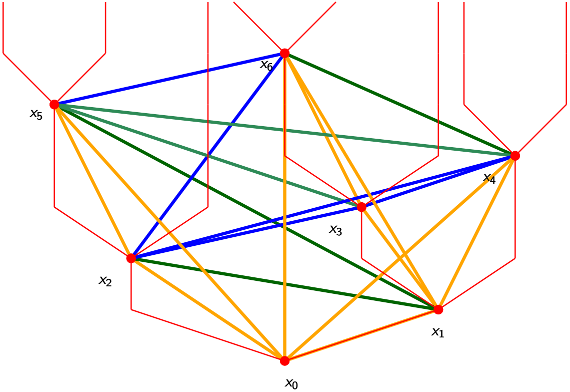

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Big Ramsey Degrees of $(\mathbb{Q}, \leq)$

In late 1960's Laver developed method of finding copies of $\mathbb{Q}$ in $\mathbb{Q}$ with bounded number of colours using Milliken's tree theorem.

Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q}, \leq) \longrightarrow (\mathbb{Q}, \leq)^{(O,\leq_O)}_{k,T}.$$

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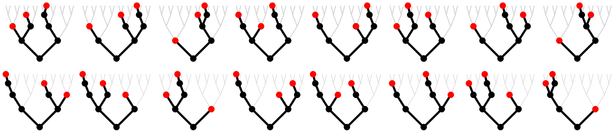
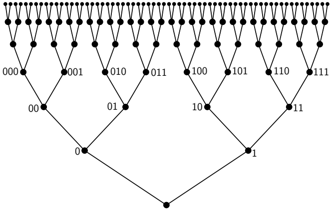


Trees (terminology)



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- A tree is a (possibly empty) partially ordered set $(T, <_T)$ such that, for every $t \in T$, the set $\{ s \in T : s <_T t \}$ is finite and linearly ordered by $<_T$. All trees considered are finite or countable.

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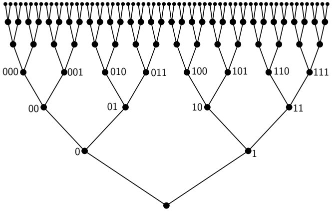

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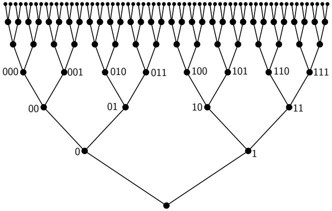

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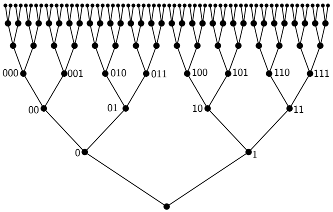

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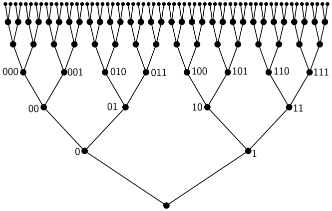

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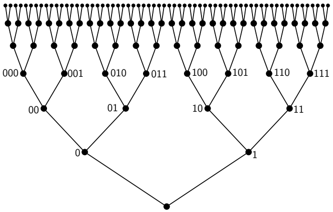

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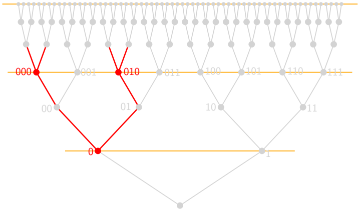
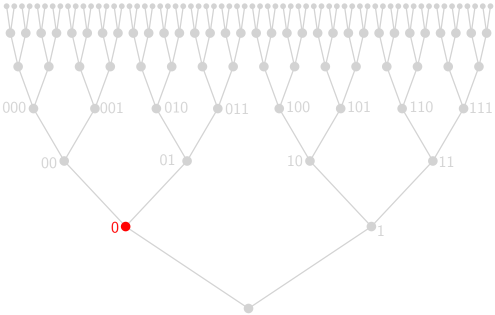
- For $s, t \in T$, the meet $s \wedge_T t$ of s and t is the largest $s' \in T$ such that $s' \leq_T s$ and $s' \leq_T t$.
- The height of T, denoted by $h(T)$, is the minimal natural number h such that $T(h) = \emptyset$. If there is no such number h, then we say that the height of T is ω.

Subtrees and strong subtrees



- A subtree of a tree T is a subset $S \subseteq T$ viewed as a tree equipped with the induced partial ordering.
- Given a tree T and nodes $s, t \in T$ we say that s is a successor of t in T if $t \leq_T s$.
- The node s is an immediate successor of t in T if $t <_T s$ and there is no $s' \in T$ such that $t <_T s' <_T s$.
- Node with no successors is leaf.

Strong subtree

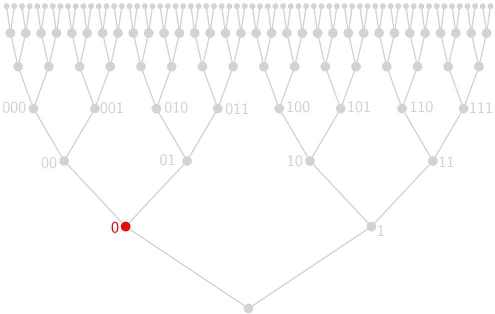


Definition

Let T be rooted tree. Nonempty $\mathbf{S} \subseteq \mathbf{T}$ is a strong subtree of T of height $n \in \omega + 1$ if:

1. S is closed for meets. (In particular, S is rooted.)

Strong subtree



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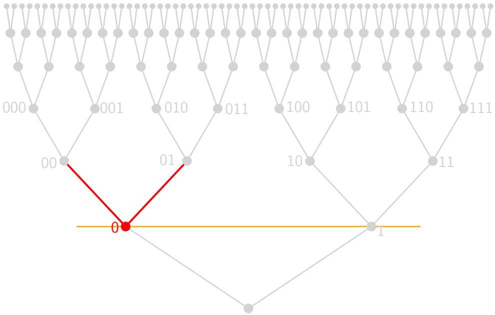

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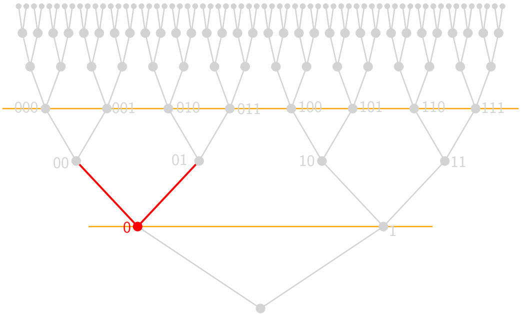

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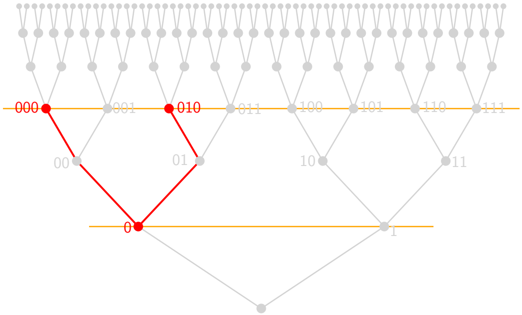

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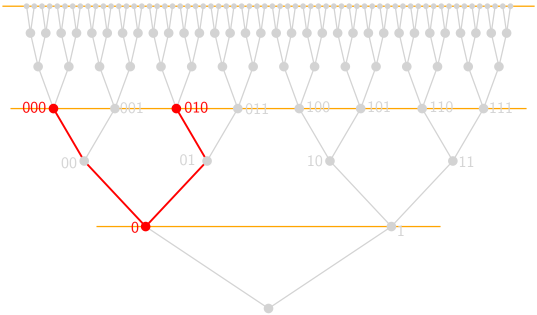

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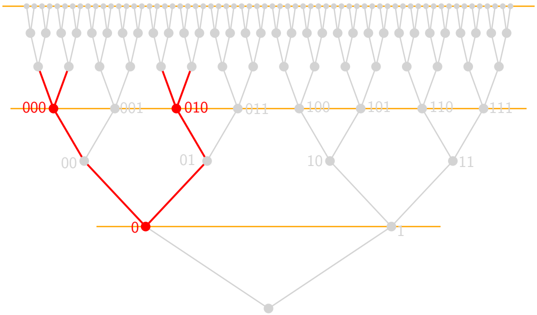

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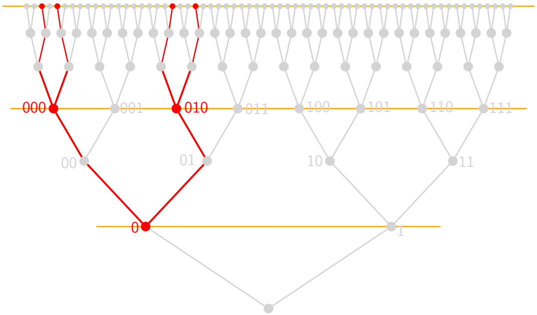

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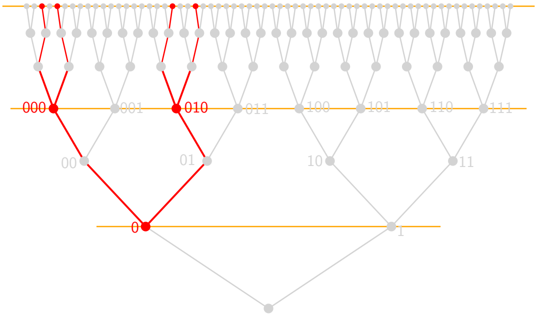

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3. S is level preserving: Every level of S is a subset of some level of T.
4. S has height n.

Ramsey-type theorem for strong subtrees

Let T be a tree and $k \in \omega + 1$. We use $\mathrm{Str}_k(T)$ to denote the set of all strong subtrees of T of height k.

Theorem (Milliken 1979)

For every rooted finitely branching tree T with no leaves, every $k \in \omega$ and every finite colouring of $\mathrm{Str}_k(T)$ there is $S \in \mathrm{Str}_\omega(T)$ such that the set $\mathrm{Str}_k(S)$ is monochromatic.

Ramsey-type theorem for strong subtrees

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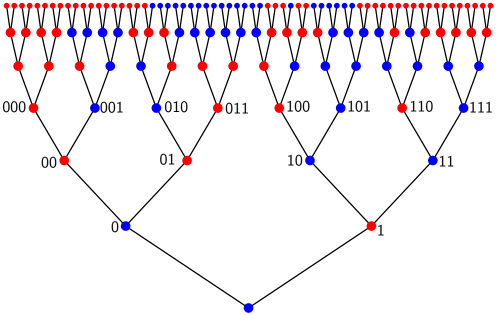

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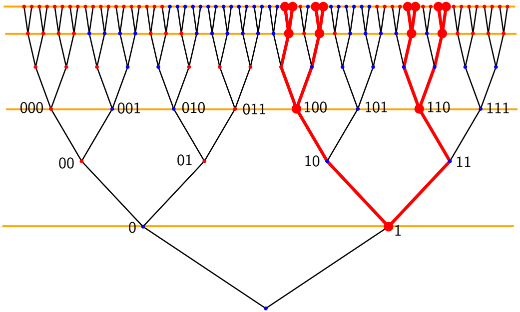

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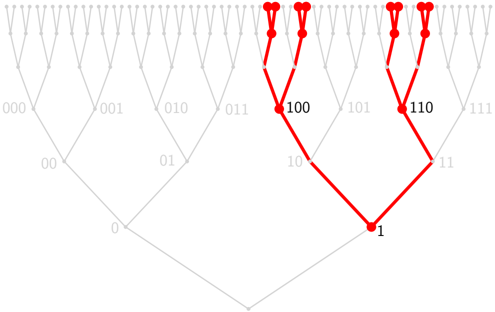

Notice that for regularly branching tree the strong subtree is isomorphic to the original tree.

Big Ramsey degrees using Milliken tree theorem

We aim to prove:

Theorem (Laver, late 1969)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1}:(\mathbb{Q},\leq)\longrightarrow(\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

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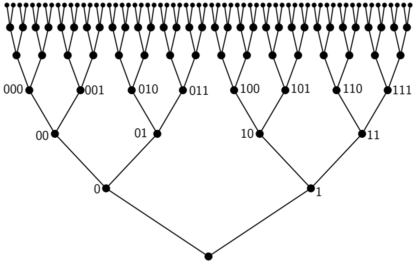

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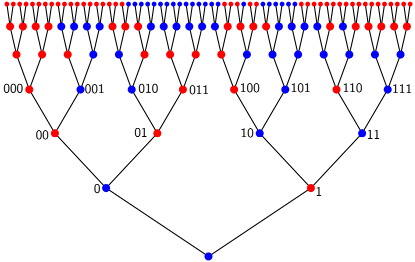

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4. Strong subtree is isomorphic to the original binary tree and also isomorphic to $(\mathbb{Q}, \leq)$ when ordered lexicographically
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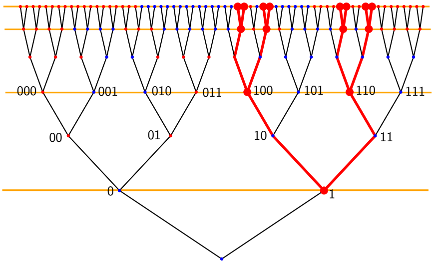

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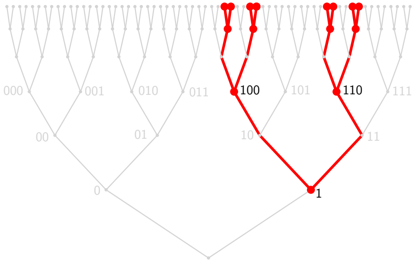

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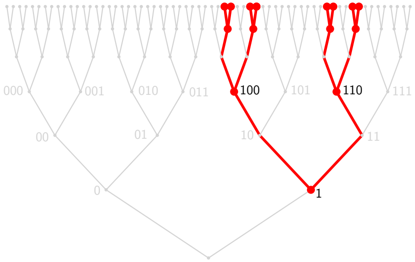
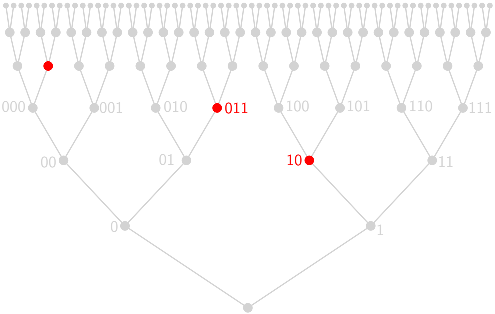


If $|O| = n > 1$ we transfer colourings of n-tuples of nodes to colouring of strong subtrees.

Envelopes of subsets



- Given a subset X of tree T, an envelope is any strong subtree S of T containing X.
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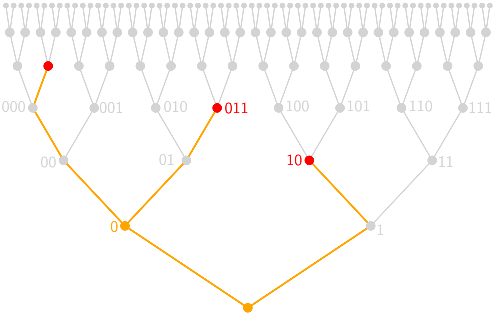

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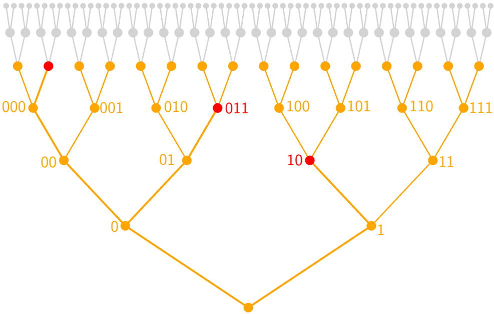

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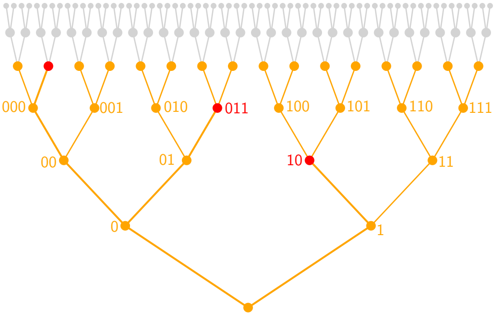

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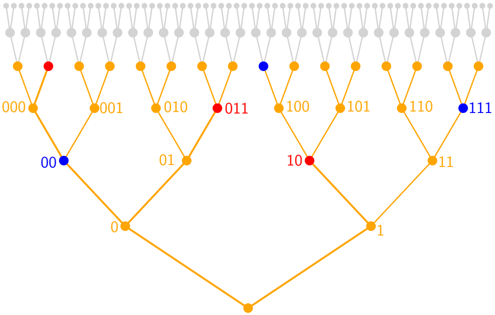

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Multiple choices of X may lead to a same envelope. We speak of different embedding types within a given envelope.

Big Ramsey degrees using Milliken tree theorem

Now we can finish proof of:

Theorem (Laver, late 1969)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1}:(\mathbb{Q},\leq)\longrightarrow(\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

Proof.

1. Fix $(O,\leq_O)\in\mathcal{O}$ and put $n=|O|$.
2. $T(n)$ is the number of embedding types of n-tuples in the binary tree.
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4. For each embedding type construct colouring of envelopes and pass to a monochromatic subtree by the application of Milliken tree theorem.
5. The resulting copy will have at most $T(n)$ different colours

□

Big Ramsey degrees of $(\mathbb{Q}, \leq)$ are finite!

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Definition (Devlin types)

$A \subseteq 2^{<\omega}$ is a Devlin embedding type iff it is an antichain and for every $0 \leq \ell < \max_{a \in A} |a|$ precisely one of the following happens:

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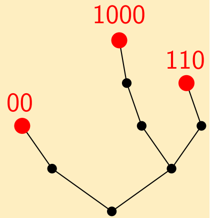

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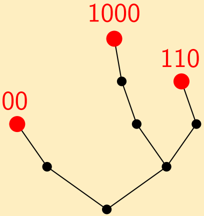

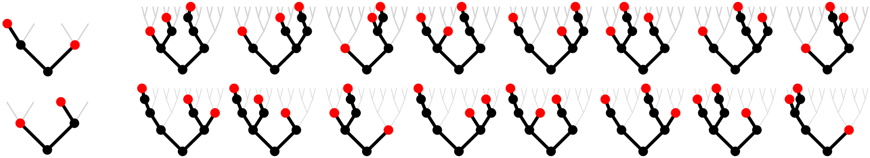

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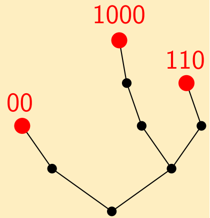

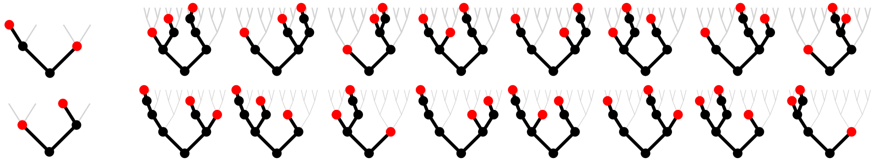

Fun fact: Number of Devlin types of size n is

$$t_n = \sum_{\ell=1}^{n-1} \binom{2n-2}{2\ell-1} t_\ell \cdot t_{n-\ell} \text{ with } n_1 = 1$$

This is well known sequence (of the odd tangent numbers).

The characterisation of the big Ramsey degrees

Theorem (Devlin, 1979)

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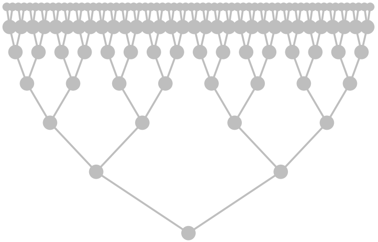

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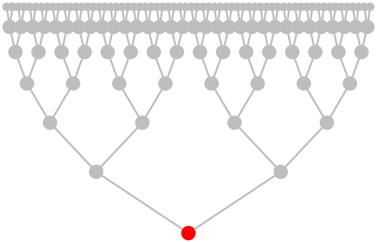

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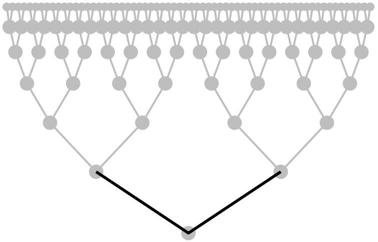

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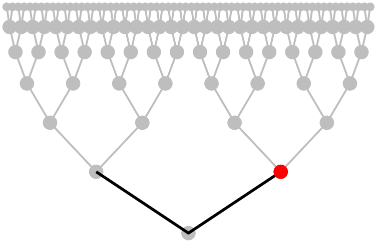

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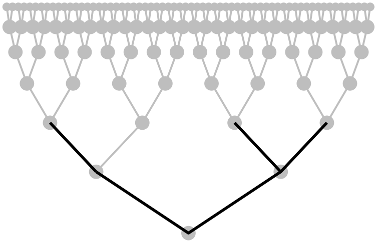

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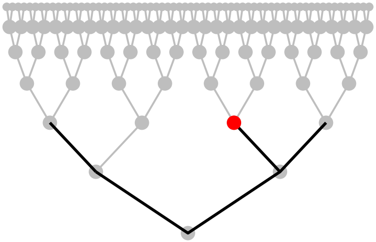

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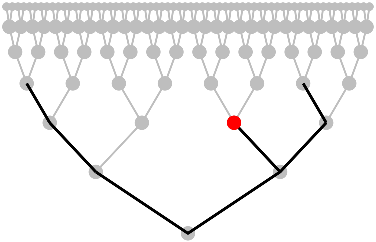

The characterisation of the big Ramsey degrees

Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q},\leq) \longrightarrow (\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

Where minimal T satisfying the statement above is the number of Devlin types of size $|O|$

Lemma

There exists Devlin type representing $(\mathbb{Q},\leq)$ (with "left to right" order of the binary tree).

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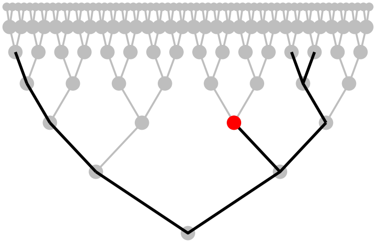

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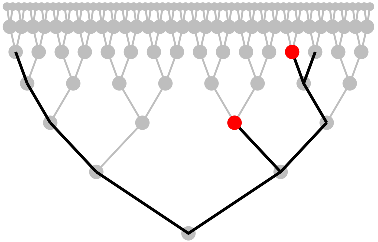

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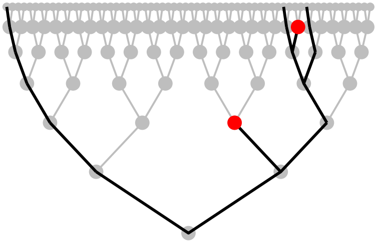

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If $B \subset A$ is a subset of Devlin type A, then the embedding type of B inside every minimal envelope is a Devlin type.

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Recall that in Devlin type on every is either leaf or branching. Minimal envelope will include all those levels where branching or leaf of B happens and skip all others. □

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We thus obtain an upper bound: $T(|O|)$ is at most the number of Devlin types.

The lower bound

Lemma

Let A be a Devlin type representing $(\mathbb{Q}, \leq)$ *and* $B \subseteq A$ *a copy of* $(\mathbb{Q}, \leq)$ *then there exists a* $C \subseteq B$ *whose embedding type (inside a minimal envelope) is A.*

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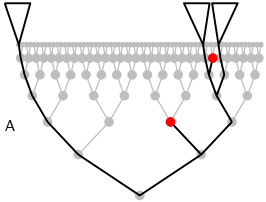
Proof by a slide-show.

□



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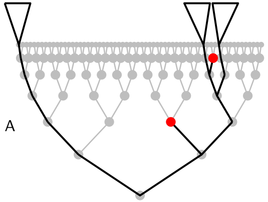

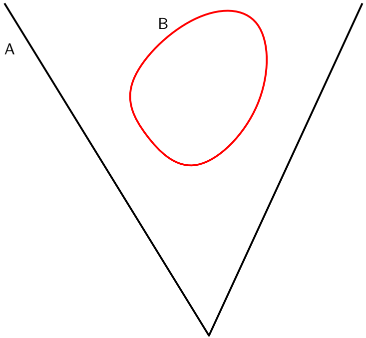

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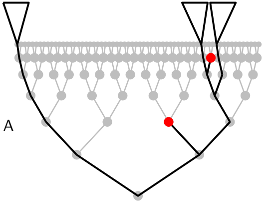
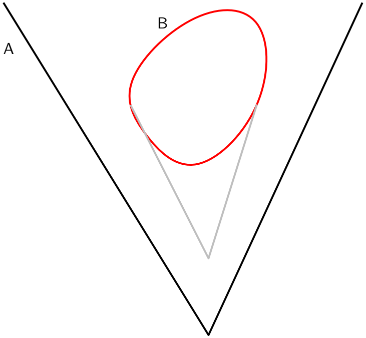
Proceed by induction on the individual branching/leaf events of A.
First produce meet of B. A starts with a branching. Place the root of A to this meet. □





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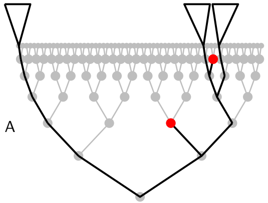

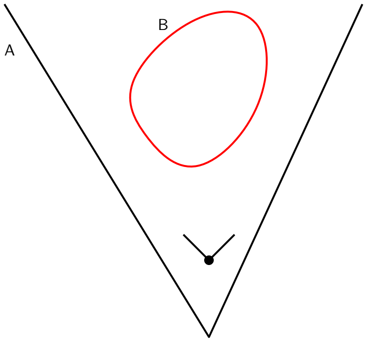

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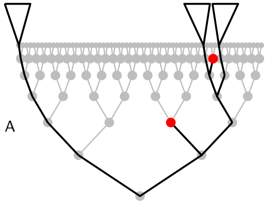
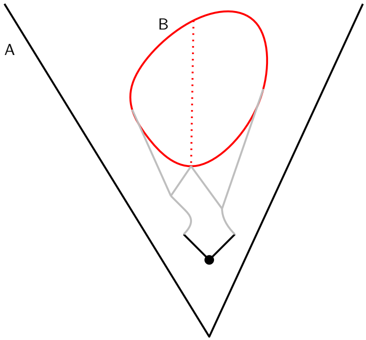
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of *A*.
The meet splits leafs of *B* into two infinite intervals. Each with meet above its son. □





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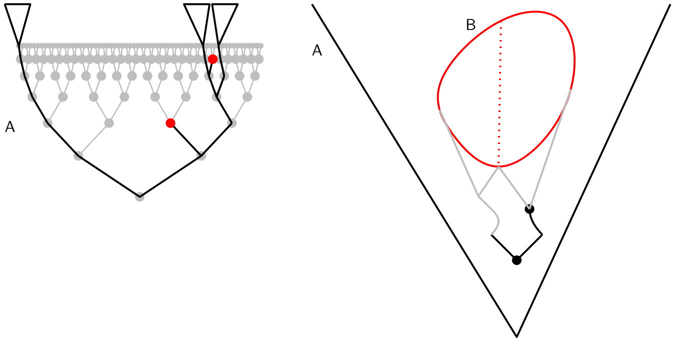
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of A.
Use corresponding meet to realize 2nd branching of A. □



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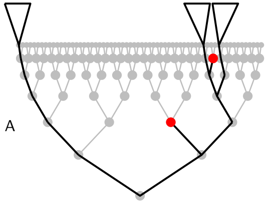
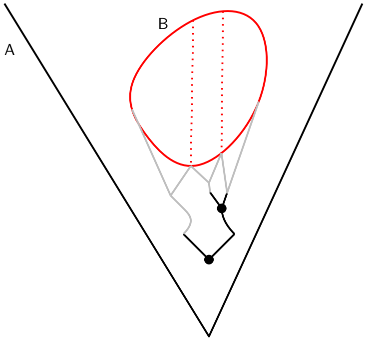
Proceed by induction on the individual branching/leaf events of A.
B further subdivides into intervals. □





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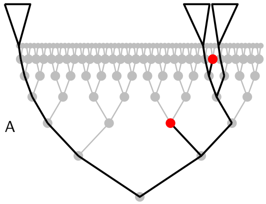
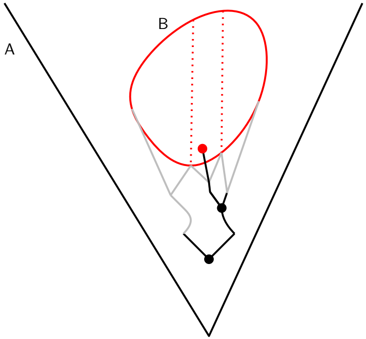
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of A.
Continue analogously with next branching. □





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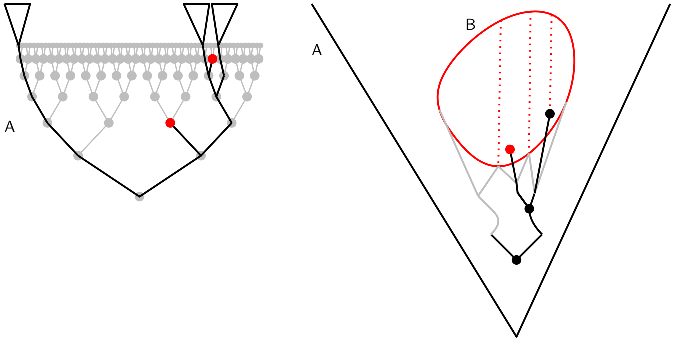
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of *A*.
Place the first leaf of *A* into corresponding interval. □



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Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of A.
...blablabla...proof done. □

Victory!

We characterised the big Ramsey degrees of rationals and gave a closed-form formula. This shows that the upper bound proof is best possible. It also has applications to topological dynamics

Kechris–Pestov–Todorčević: Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups

Zucker: Big Ramsey degrees and topological dynamics

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- We denote by $\mathcal{G}$ the class of all finite graphs.

Theorem

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{T=T'(\mathbf{A})\in\omega}\forall_{k\geq 1} : \mathbf{R} \longrightarrow (\mathbf{R})^{\mathbf{A}}_{k,T}.$$

This theorem was published by Sauer in 2006 and also appears in Todorčević' Introduction to Ramsey spaces. Values of $T'(\mathbf{G})$ were characterised by Laflamme–Sauer–Vuksanović in 2010.

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A finitary version is (probably more) famous!

Theorem (Nešetřil–Rödl 1977, Abramson–Harington 1978)

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{t=t(\mathbf{A})\in\omega}\forall_{\mathbf{B}\in\mathcal{G},k\geq 1}\exists_{\mathbf{C}} \in \mathcal{G} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}.$$

Understanding the unavoidable colourings

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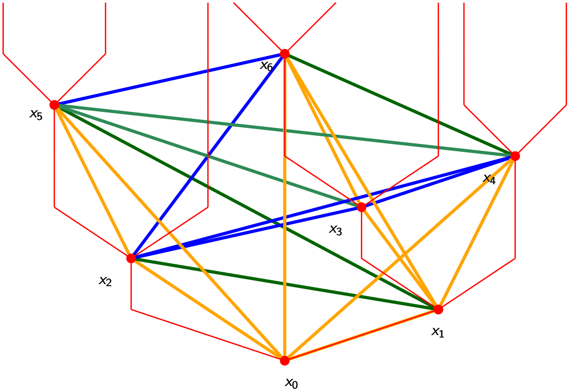
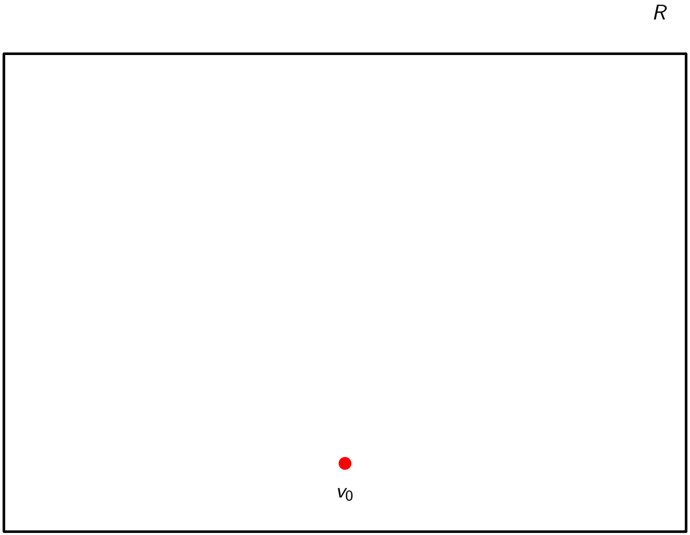


For $(\mathbb{Q}, \leq)$ we have the Sierpiński colourings. Can we do something similar for the Rado graph?

Understanding the unavoidable colourings of Rado graphs

R

Understanding the unavoidable colourings of Rado graphs



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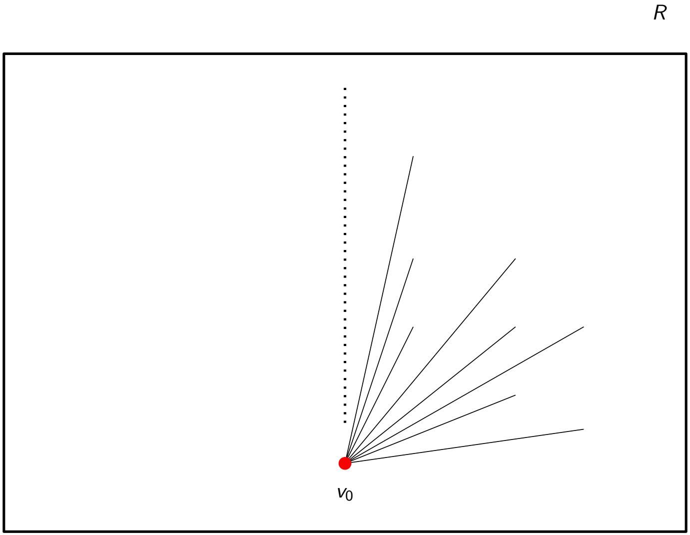

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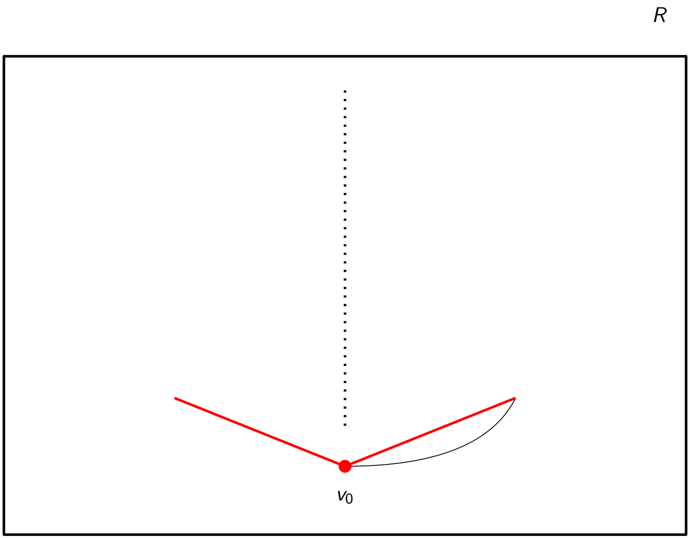

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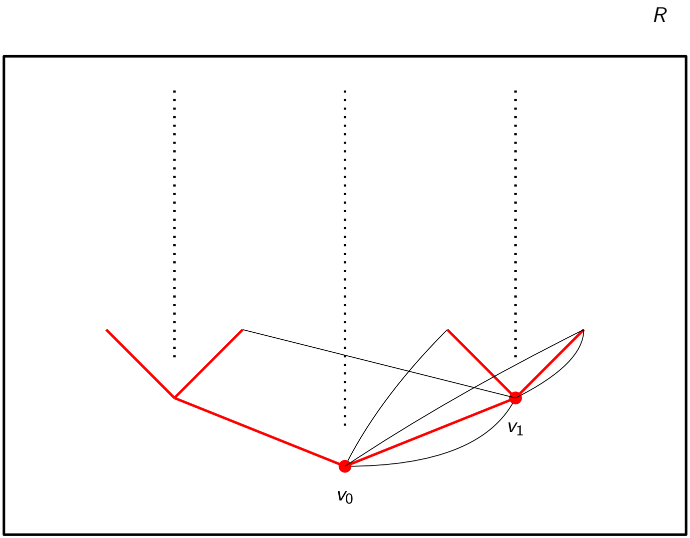

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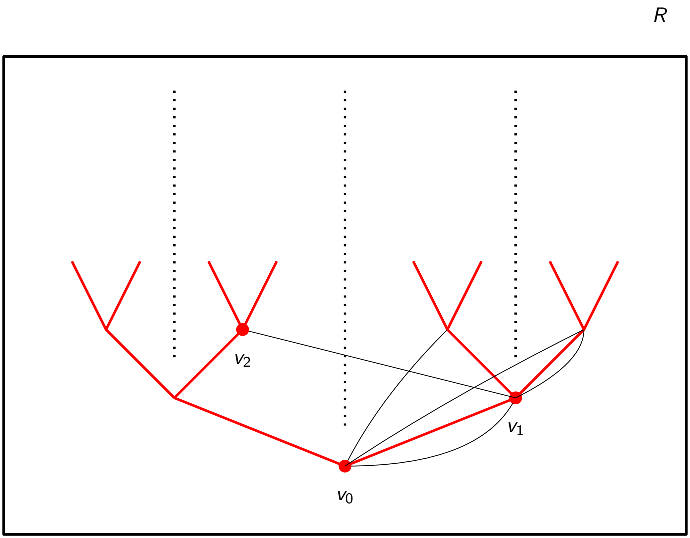

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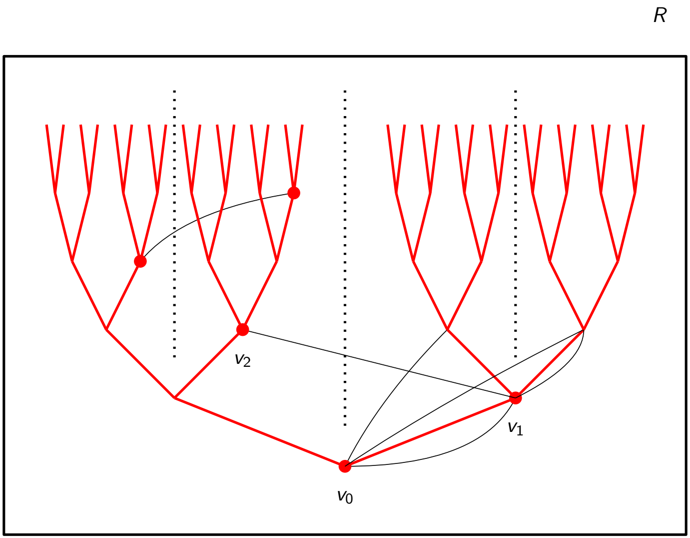

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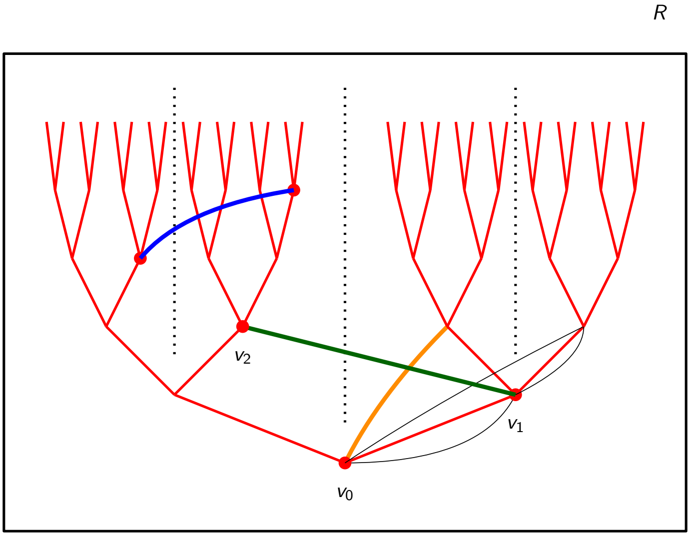
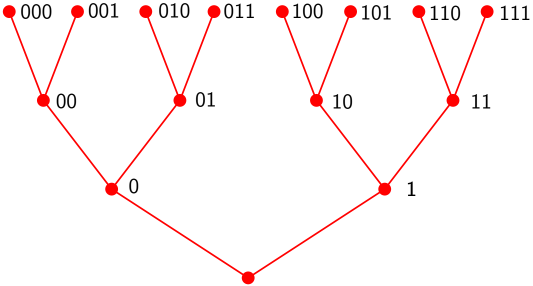


Passing number graph

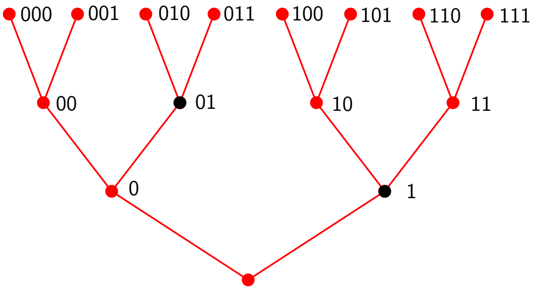


Definition (Graph **G**)

We will consider graph **G**:

1. Vertices: $2^{<\omega}$
2. Vertices $a, b \in 2^{<\omega}$ satisfying $|a| < |b|$ forms and edge if and only if $b(|a|) = 1$.
3. There are no other edges.

Passing number graph



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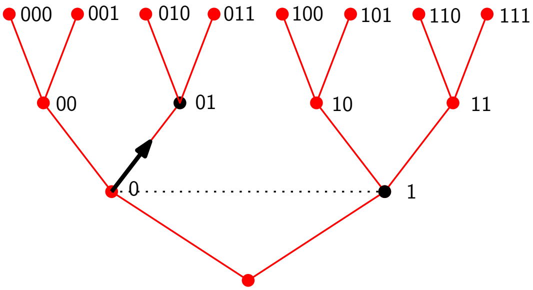

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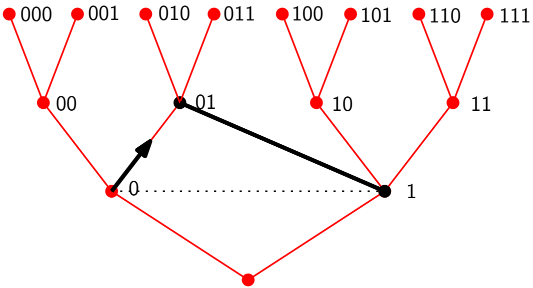

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***G** is universal: the Rado graph **R** embeds to **G**.*

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Assume that the vertex set of **R** is ω. The vertex $i \in \omega$ then corresponds to a sequence a of length i with $a(j) = 1$ if and only if $i \sim j$. $\square$

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We thus can repeat precisely the same proof as before to obtain the upper bound on big Ramsey degrees.

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Lower bounds needs a bit more care.

Thank you for the attention

Most we covered today is in S. Todorčević, Introduction to Ramsey spaces

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