Ramsey theorems for classes of structures with functions and relations

Jan Hubička

Department of Applied Mathematics
Charles University
Prague

Joint work with David Evans, Matěj Konečný and Jaroslav Nešetřil

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Ramsey theorem for finite relational structures

Let L be a purely relational language with binary relation $\leq$.

Denote by $\overrightarrow{Rel}(L)$ the class of all finite L-structures where $\leq$ is a linear order.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)}\exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

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Theorem (Ramsey Theorem, 1930)

$$\forall_{n,p,k\geq 1}\exists_N : N \longrightarrow (n)^p_k.$$

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$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.

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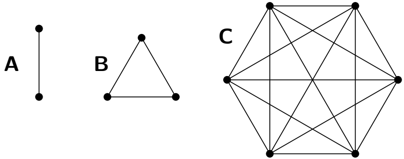

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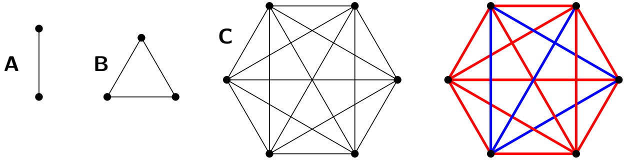
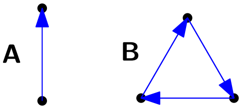


Order is necessary



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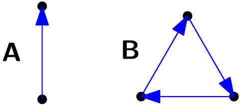
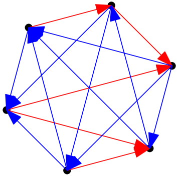


Vertices of **C** can be linearly ordered and edges coloured accordingly:

- If edge is goes forward in linear order it is red
- blue otherwise.



Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Example (Linear orders — Ramsey Theorem, 1930)

The class of all finite linear orders is a Ramsey class.

Example (Structures — Nešetřil-Rödl, 76; Abramson-Harrington, 78)

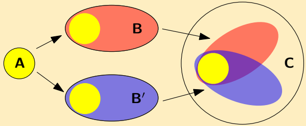
For every relational language L, class of all finite ordered L-structures is a Ramsey class.

Example (Partial orders — Nešetřil-Rödl, 84; Paoli-Trotter-Walker, 85)

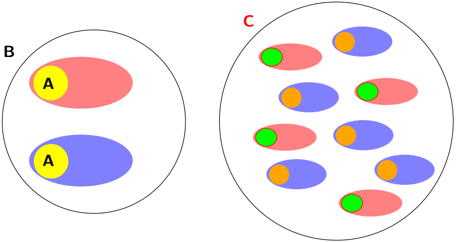
The class of all finite partial orders with linear extension is Ramsey.

Ramsey classes are amalgamation classes

Definition (Amalgamation)



Nešetřil, 80's: Under mild assumptions Ramsey classes have amalgamation property.

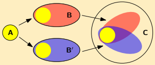


Fraïssé limits

Definition (Amalgamation class)

A class $\mathcal{K}$ of finite relational structures is called an amalgamation class if the following conditions hold:

1. $\mathcal{K}$ is hereditary (closed under substructures).
2. $\mathcal{K}$ is closed under isomorphisms.
3. $\mathcal{K}$ has only countably many mutually non-isomorphic structures.
4. $\mathcal{K}$ has the amalgamation property



A structure **A** is homogeneous if every isomorphism of two induced finite substructures of **A** can be extended to an automorphism of **A**.
Age(**U**) is the class of all finite structures isomorphic to a substructure of **U**.

Theorem (Fraïssé)

A class $\mathcal{K}$ of finite structures is the age of a countable homogeneous structure **G** *if and only if $\mathcal{K}$ is an amalgamation class.*

Nešetřil's Classification Programme, 2005

Classification Programme

Ramsey classes	$\Longrightarrow$	amalgamation classes
$\Uparrow$		$\Downarrow$
expansions of homogeneous	$\Longleftarrow$	homogeneous structures

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Let L' be language containing language L. A expansion of L-structure $\mathbf{A}$ is L'-structure $\mathbf{A}'$ on the same vertex set such that all relations/functions in $L \cap L'$ are identical.

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Theorem (Nešetřil, 1989)

All homogeneous graphs have Ramsey expansion.

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words

Milliken tree theorem: C-relations

Ramsey's theorem: rationals

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Permutations

Equivalences

Ramsey's theorem: rationals

Product arguments

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words → Boolean algebras → Semilattices

Milliken tree theorem: C-relations

Permutations

Interval graphs

Equivalences → Cyclic orders

Ramsey's theorem: rationals

Product arguments   Interpretations

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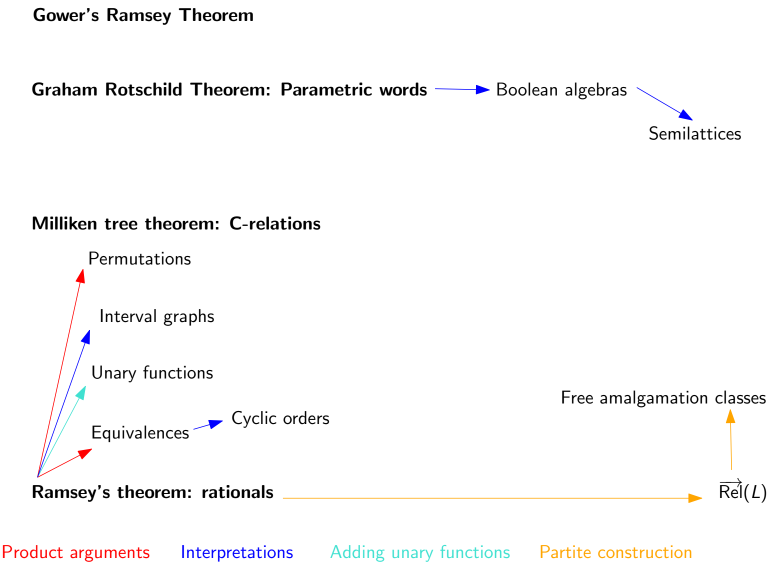
Adding unary functions


Gower's Ramsey Theorem
Graham Rotschild Theorem: Parametric words
Boolean algebras
Semilattices
Milliken tree theorem: C-relations
Permutations
Interval graphs
Unary functions
Equivalences
Cyclic orders
Free amalgamation classes
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$\overrightarrow{\mathrm{Rel}}(L)$
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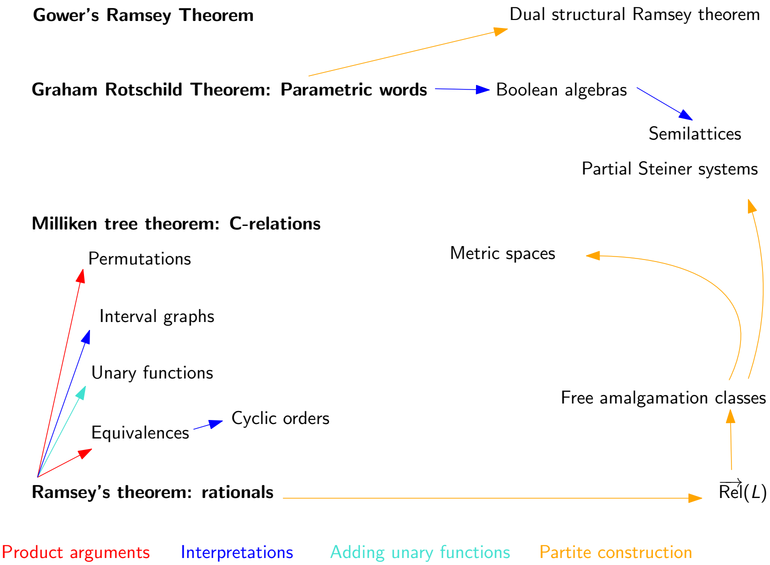
Partite construction


Gower's Ramsey Theorem
Dual structural Ramsey theorem
Graham Rotschild Theorem: Parametric words
Boolean algebras
Semilattices
Partial Steiner systems
Milliken tree theorem: C-relations
Permutations
Metric spaces
Interval graphs
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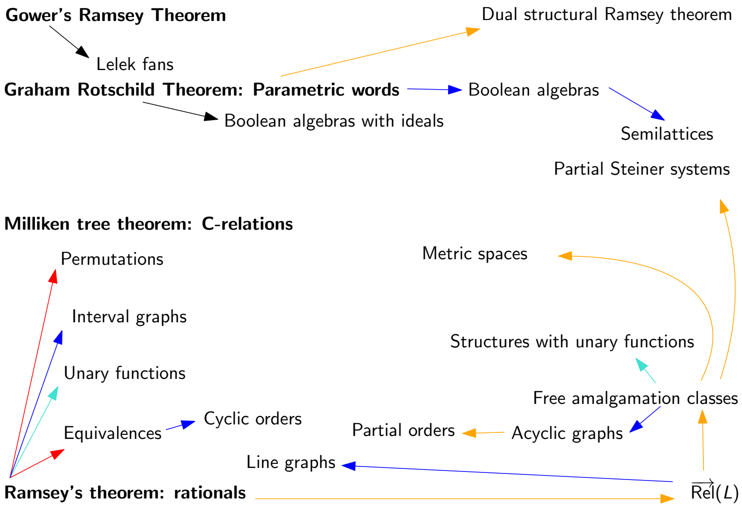
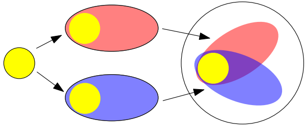
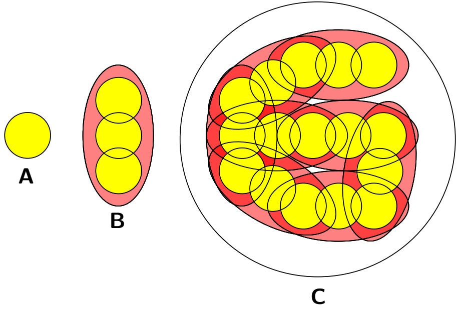


Product arguments | Interpretations | Adding unary functions | Partite construction

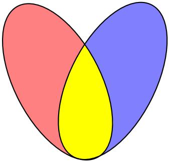
Why Ramsey objects are hard to construct?



The Nešetřil-Rödl partite construction of Ramsey object demands more complicated (multi)amalgamations.



Systematic approach



free amalgamation
(graphs, triangle free graphs)

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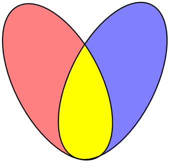

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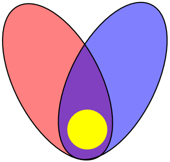

amalgamation with closure
(structures with functions, Steiner systems)

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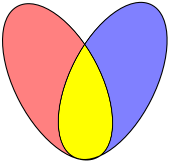

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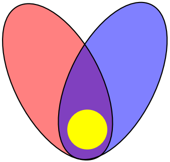

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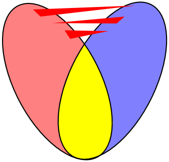

strong amalgamation
(orders, metric spaces)

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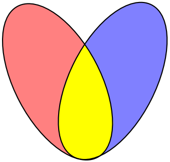

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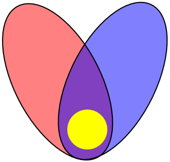

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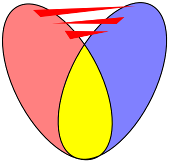

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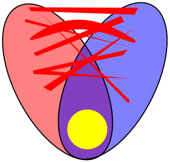

general case
(boolean algebras, groups, matroids)

Free amalgamation classes

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Definition (classes with free ordering)

1. Given a language L, $\overrightarrow{L}$ expands L by binary relational symbol $\leq$.
2. Given an L-structure $\mathbf{A}$, an ordering of $\mathbf{A}$ is an $\overrightarrow{L}$-structure expanding $\mathbf{A}$ by an arbitrary linear ordering $\leq_{\mathbf{A}}$ of the vertices.
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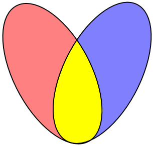

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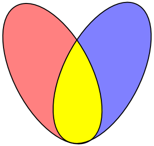

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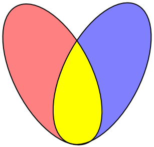

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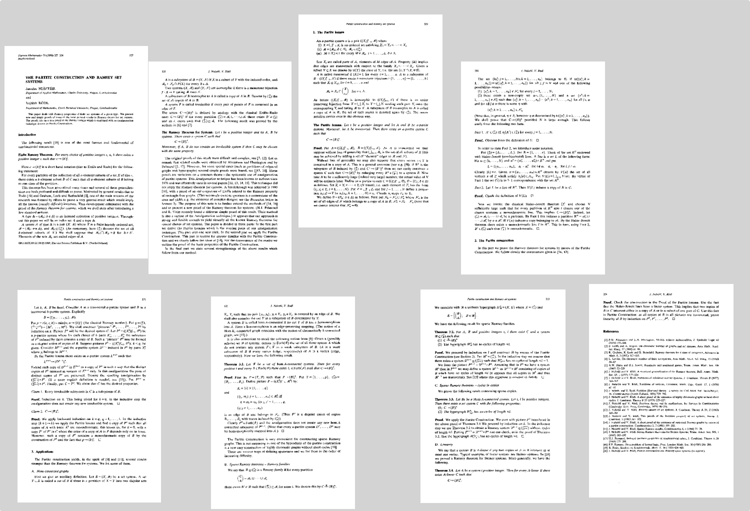
Corollary (Ordering (expansion) property)

Let L be a relational language and $\mathcal{K}$ be a free amalgamation class of L-structures containing only one isomorphism type of structure with 1 vertex. Then for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ so that every ordering of $\mathbf{B}$ contains every ordering of $\mathbf{A}$.

Nešetřil-Rödl: The Partite Construction and Ramsey Set Systems (1989)



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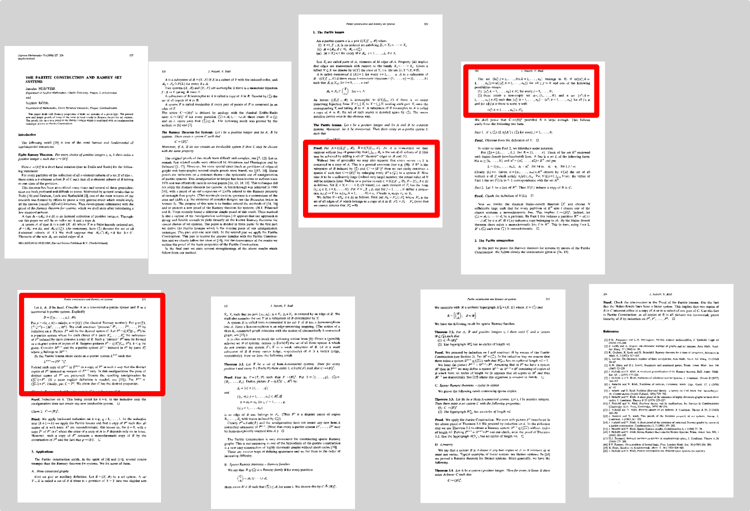

Applications

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    2. K_n-free graph and complements

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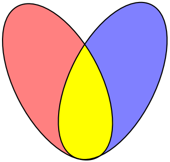
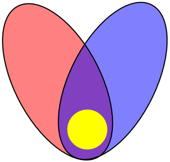
   4. ...
3. k-colourable graphs, or generally CSP(**H**): the class of all finite structures with homomorphism to **H**.
4. classes of digraphs with no homomorphic image of a given oriented tree **T**. (follows from graph duality characterisation by Nešetřil and Tardif, 1999)
5. ...

Systematic approach



free amalgamation
(graphs, triangle free graphs)
All done in 70's



amalgamation with closure
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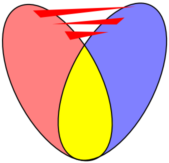

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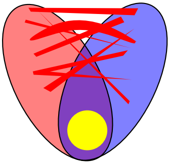

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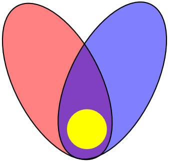

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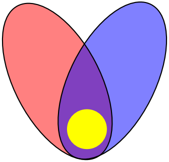

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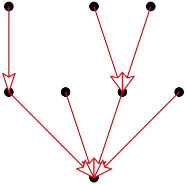
Example (Forests)

- Let $\mathcal{F}$ be the class of all finite structure with one unary function which represent a forest: $F(\text{son}) = \text{father}$.
- No ordering property: forests can always be ordered level-wise.



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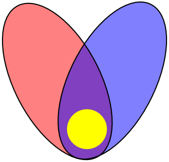
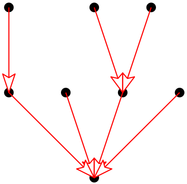


1. We consider languages with both relations and functions.
2. To make free amalgamation meaningful for non-unary functions we consider partial functions.

Example (Forests)

- Let $\mathcal{F}$ be the class of all finite structure with one unary function which represent a forest: $F(\text{son}) = \text{father}$.
- No ordering property: forests can always be ordered level-wise.



Theorem (Evans, H., Nešetřil, 2017+)

Let L be a language and $\mathcal{K}$ be a free amalgamation class of L-structures. Then there exists Ramsey amalgamation class $\mathcal{K}^+ \subseteq \overrightarrow{\mathcal{K}}$ of admissible orderings such that for every $\mathbf{A} \in \mathcal{K}$ there exists ordering $\overrightarrow{\mathbf{A}} \in \mathcal{K}^+$ and $\mathbf{B} \in \mathcal{K}$ so that every ordering of $\mathbf{B} \in \mathcal{K}^+$ contains every ordering of $\mathbf{A} \in \mathcal{K}^+$.

Hrushovski class has optimal Ramsey expansion

We consider class $\mathcal{C}_F$ of (special) 2-orientable graphs from David Evans talk which has ω-categorical limit built by Hrushovski predimension construction.

Theorem (Evans, H., Nešetřil 2018+)

There exists (non-precompact) Ramsey expansion $\mathcal{G}_F$ of $\mathcal{C}_F$ with adds:

1. *functions representing closures which can be realised by fine 2-orientation,*
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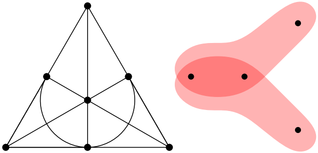
1. $\mathcal{C}_F$ with special substructures is not a free amalgamation class in our sense! (new functions needs to be added into free amalgamation to special substructures of amalgamation).
2. after fixing 2-orientation the class becomes a free amalgamation class (now special closures have easy combinatorial meaning).
3. oriented class is not optimal: orientation needs to be forgotten again!
4. resulting class is a free amalgamation class and Ramsey property follows.

Applications

1. Partial Steiner systems
(Bhat, Nešetřil, Reiher, Rödl, 2016)



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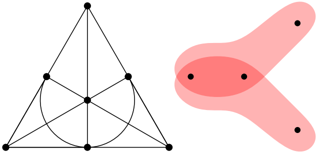

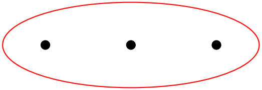

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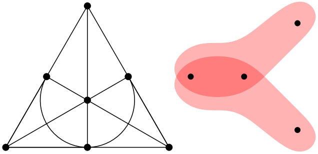

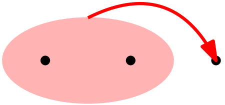

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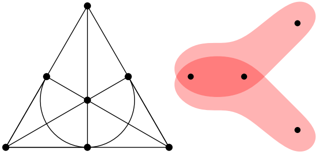

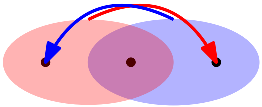

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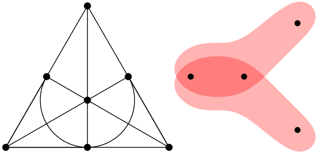

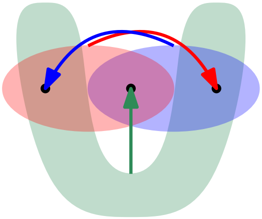

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      (Nguyen Van Thé, 2009)
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5. All known Cherlin-Shelah-Shi classes (classes of graphs defined by forbidden monomorphisms from a given graph G with ω-categorical universal graph) (for bowtie free graphs Nešetřil, H. 2018)
6. Unary functions (Sokić, 2016)

Unary functions are easy!

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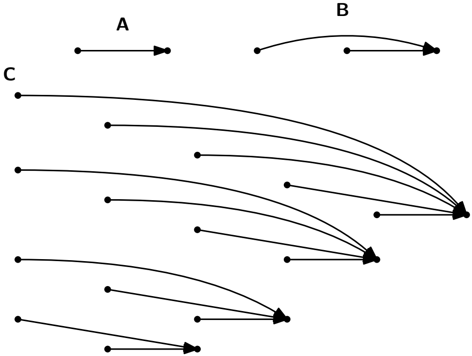

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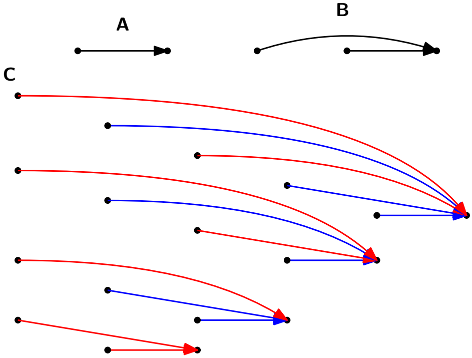

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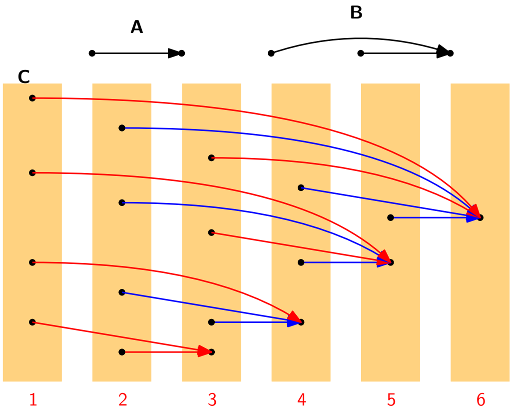

$$6 \longrightarrow (|B|)_2^{|A|}$$

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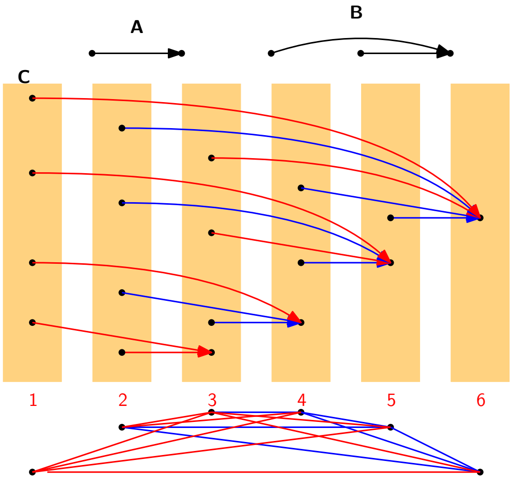

$$6 \longrightarrow (|B|)^{|A|}_2$$

Every $|B|$-tuple of parts corresponds to a copy of B

Every $|A|$-tuple corresponds to at most one copy of A

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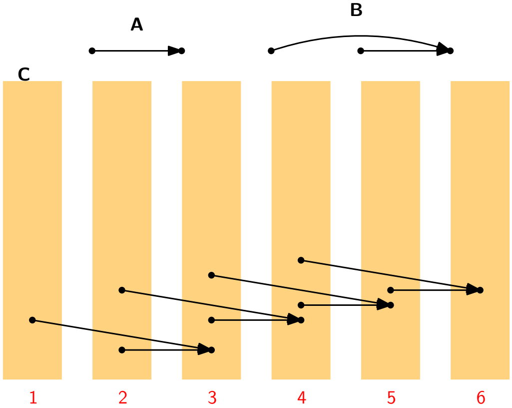

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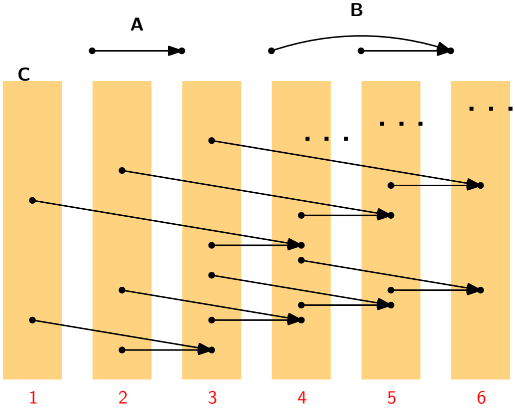

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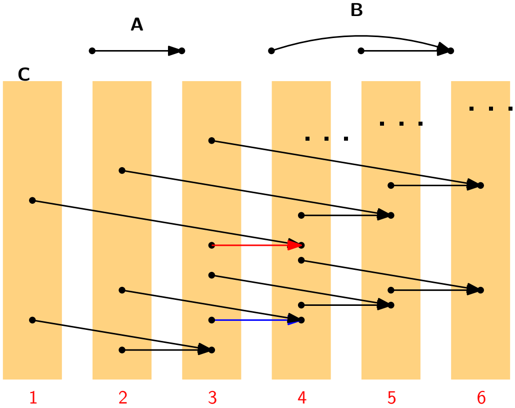

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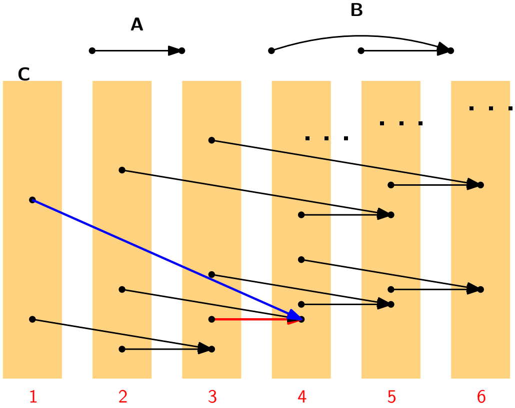

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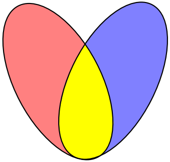
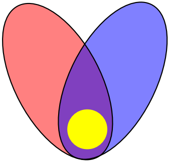
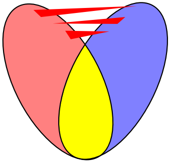
Systematic approach



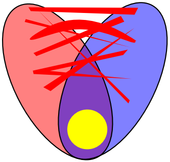
free amalgamation
(graphs, triangle free graphs)
All done in 70’s



amalgamation with closure
(structures with functions, Steiner systems)
All done last year!



strong amalgamation
(orders, metric spaces)



general case
(boolean algebras, groups, matroids)

Structural condition

Theorem (H.-Nešetřil, 2016)

Let L be language with relations and (partial) functions. Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a strong amalgamation subclass of $\mathcal{R}$. If $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ then $\mathcal{K}$ is Ramsey.

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Schematically

Recall:

Ramsey classes	$\Longrightarrow$	amalgamation classes
$\Uparrow$		$\Downarrow$
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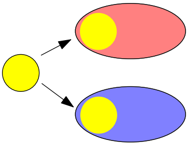
We get:

strong amalgamation + order + local finiteness $\Longrightarrow$ Ramsey

What is local finiteness?

Multiamalgams as structures with holes

Representing multiamalgams as “completion of structures with holes”:



An L-structure $\mathbf{A}$ is irreducible if it can not be created as a free amalgamation of its two proper substructures.

Amalgamation of irreducible structures is

1. free amalgamation,
2. completion.

Definition

Irreducible structure $\mathbf{C}'$ is a completion of $\mathbf{C}$ if it has the same vertex set and every irreducible substructure of $\mathbf{C}$ is also (induced) substructure of $\mathbf{C}'$.

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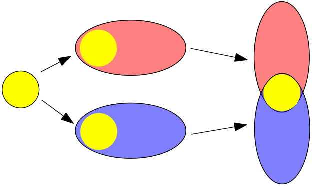

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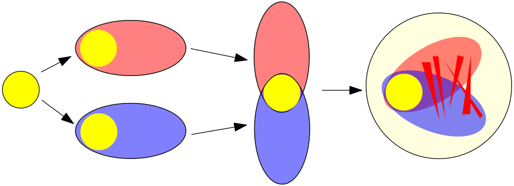

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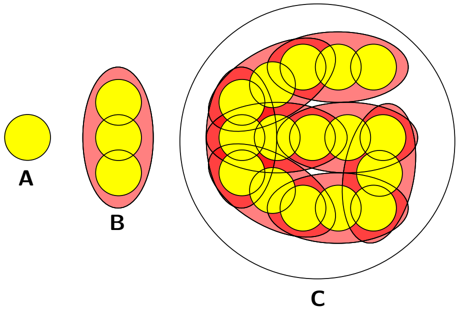
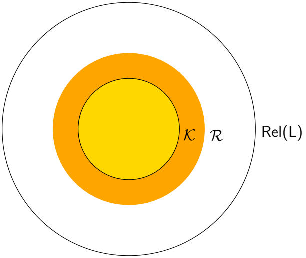
Taming of the multiamalgamation



Multiamalgamation which locally is amalgamation

Intuition

$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.



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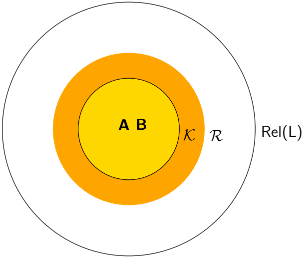

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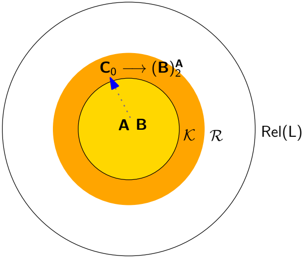

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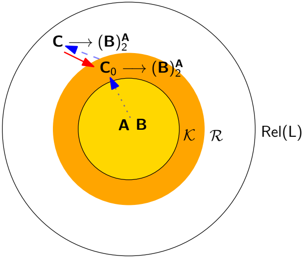

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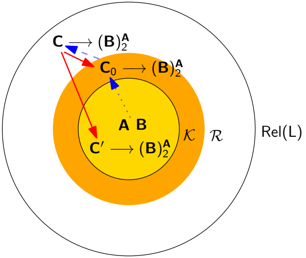

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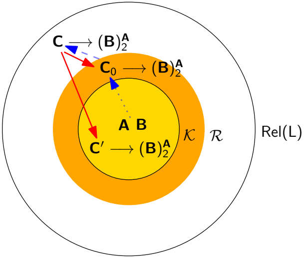
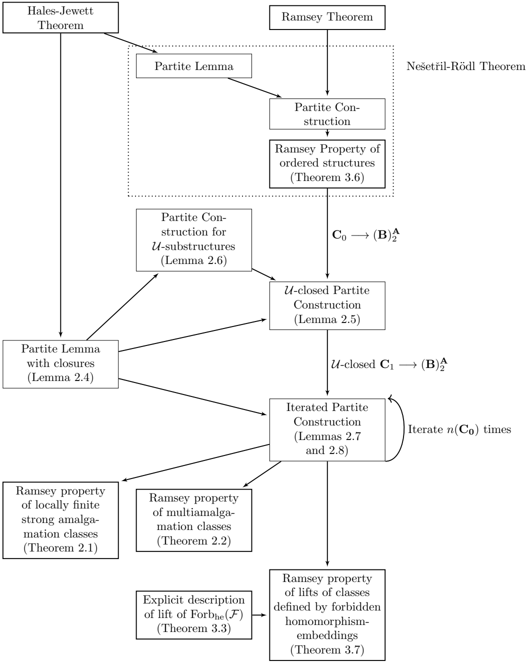


Definition

Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ there is $n = n(\mathbf{C}_0)$ such that every structure $\mathbf{C}$ has completion in $\mathcal{K}$ providing that it satisfies the following:

1. there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$
2. every substructure of $\mathbf{C}$ with at most n vertices has a completion in $\mathcal{K}$.

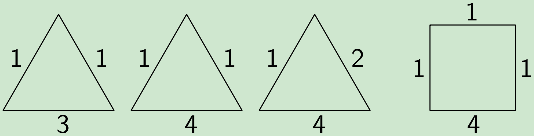
homomorphism-embedding is a homomorphism which is an embedding on every irreducible substructure.


Hales-Jewett Theorem
Ramsey Theorem
Partite Lemma
Nešetřil-Rödl Theorem
Partite Construction
Ramsey Property of ordered structures (Theorem 3.6)
Partite Construction for $\mathcal{U}$-substructures (Lemma 2.6)
$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
$\mathcal{U}$-closed Partite Construction (Lemma 2.5)
Partite Lemma with closures (Lemma 2.4)
$\mathcal{U}$-closed $\mathbf{C}_1 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
Iterated Partite Construction (Lemmas 2.7 and 2.8)
Iterate $n(\mathbf{C_0})$ times
Ramsey property of locally finite strong amalgamation classes (Theorem 2.1)
Ramsey property of multiamalgamation classes (Theorem 2.2)
Explicit description of lift of $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ (Theorem 3.3)
Ramsey property of lifts of classes defined by forbidden homomorphism-embeddings (Theorem 3.7)

Locally finite subclass, an example

Example

Consider class of metric spaces with distances $\{1, 2, 3, 4\}$. Graph with edges labelled by $\{1, 2, 3, 4\}$ can be completed to a metric space if and only if it does not contain one of:



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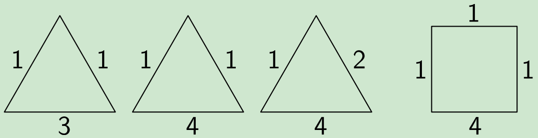

The class $\overrightarrow{\mathcal{M}}_k$ of all ordered metric spaces with integer distances at most k is Ramsey.

Locally finite subclass, an example

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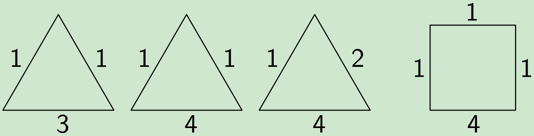


The class $\overrightarrow{\mathcal{M}}_k$ of all ordered metric spaces with integer distances at most k is Ramsey.

Theorem (Nešetřil, 2007)

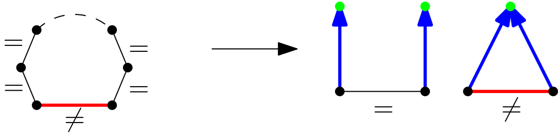
The class $\overrightarrow{\mathcal{M}}_{\mathbb{Q}}$ of all metric spaces with rational distances is Ramsey.

Special metric spaces

Theorem (H., Nešetřil, 2016+)

Every $S \subseteq \mathbb{R}$ such that S-metric spaces (using only distances in S) forms an amalgamation class this class has Ramsey expansion.

Special cases of $|S| \leq 4$ proved in Nguyen Van Thé 2010.



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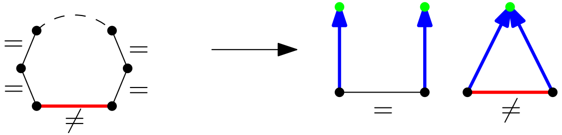

Theorem (Aranda, H., Karamanlis, Kompatscher, Konečný, Pawliuk, Bradley-Williams, 2016+)

All metrically homogeneous graphs from Cherlin's conjectured catalogue with exception of tree-like ones have precompact Ramsey expansion.
Tree-like ones have no interesting Ramsey expansion for trivial reasons.

Applications

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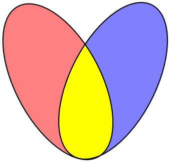
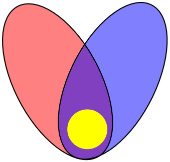
    1. metric spaces valued by partially ordered semigroup (common generalisation of structures of Cherlin's metrically homogeneous graphs and generalisations of metric spaces by Samuel Braunfeld and Gabriel Conant.)
3. All of the catalogue of homogeneous directed graphs (Jasiński, Laflamme, Nguyen Van Thé, Woodrow, 2013)
    1. Partial orders (Nešetřil-Rödl, 1984; Paoli-Trotter-Walker, 1985)
    2. Semigeneric tournament
    3. . . .

Systematic approach



free amalgamation
(graphs, triangle free graphs)
All done in 70's



amalgamation with closure
(structures with functions, Steiner systems)
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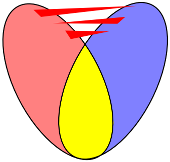
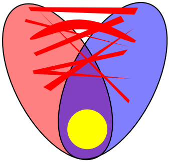


strong amalgamation
(orders, metric spaces)
Structural condition
covering all known examples



general case
(boolean algebras, groups, matroids)

General amalgamation classes

Examples following by local finiteness argument:

1. Antipodal structures
(such as those in catalogue of metrically homogeneous graphs)
2. All known Cherlin-Shelah-Shi classes
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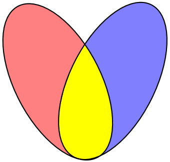
Open problems

1. graphs of girth ≥ 5
2. Steiner systems with no short odd cycles
3. Matroids of rank 3
4. ...

Systematic approach



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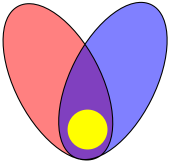

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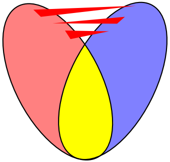

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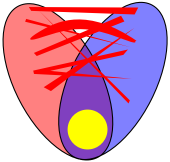
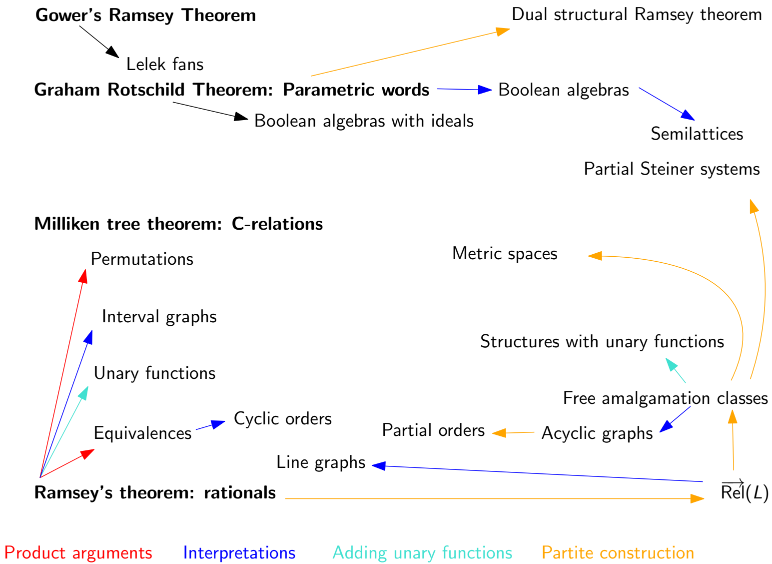


general case
(boolean algebras, groups, matroids)
Structural condition
maybe covering all known examples;
number of open problems;
some negative results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Lelek fans
Dual structural Ramsey theorem
Graham Rotschild Theorem: Parametric words
Boolean algebras
Boolean algebras with ideals
Semilattices
Partial Steiner systems
Milliken tree theorem: C-relations
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$\overrightarrow{\mathrm{Rel}}(L)$
Product arguments
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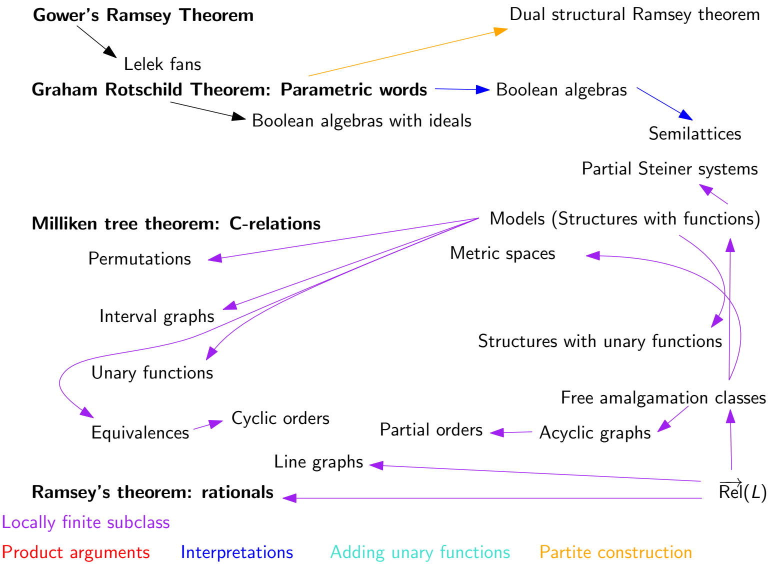
Adding unary functions
Partite construction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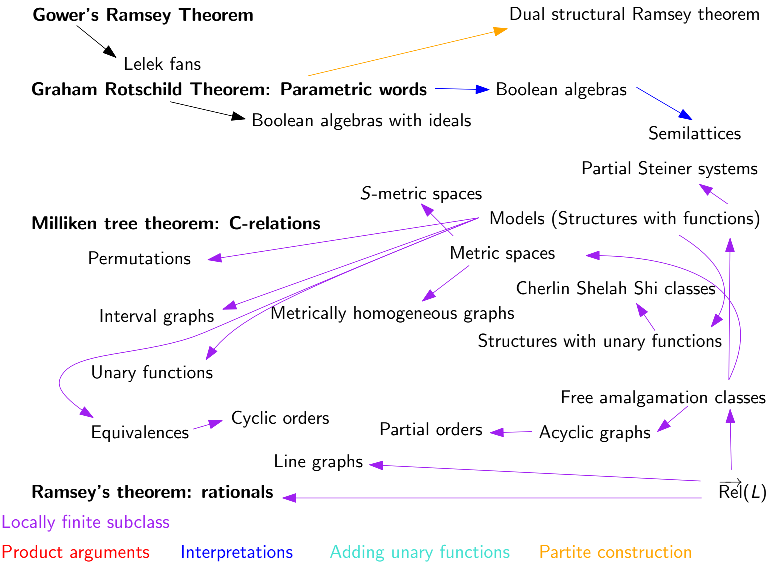
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Other “better amalgamations”

Our approach also applies to:

- Extension property for partial automorphisms (Hrushovski property)
  - Graphs (Hrushovski, 1992)
  - Relational structures, forbidden irreducible substructures (Herwig, 1998)
  - Free amalgamation classes (Hodkinson, Otto, 2003; Siniora, Solecki, 2016+)
  - Strong amalgamation classes with finitely many obstacles (Herwig, Lascar, 2000, Otto 2017+)
  - Free amalgamation classes with unary functions (Evans, H., Nešetřil 2016+)
- Stationary independence relation (Tent, Ziegler, 2013)
- Canonical independence relation (Kaplan, Simon, 2016+)

Thank you for the attention

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