Locally injective homomorphisms are universal on connected graphs

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Joint work with Jirka Fiala and Yangjing Long

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Universal relational structures

Let $\mathcal{C}$ be class of relational structures.

Definition

Relational structure $\mathbf{U}$ is **(embedding-)universal** for class $\mathcal{C}$ iff $\mathbf{U} \in \mathcal{C}$ and every structure $\mathbf{A} \in \mathcal{C}$ is induced substructure of $\mathbf{U}$.

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Example:

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- Explicit description by Rado:
  - **Vertices:** all finite 0–1 sequences $(a_1, a_2, \ldots, a_t)$, $t \in \mathbb{N}$
  - **Edges:** $\{(a_1, a_2, \ldots, a_t), (b_1, b_2, \ldots, b_s)\}$ form edge iff $b_a = 1$ where $a = \sum_{i=1}^{t} a_i 2^i$.

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In this talk we a give new one.

The homomorphism order

- Denote by $\mathcal{G}$ the class of all finite graphs.
- (Graph) **homomorphism** $f : G \to H$ is an edge preserving mapping: $\{u, v\} \in E_G \implies \{f(u), f(v)\} \in E_H$.

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- Graphs G and H are **hom-equivalent**, $G \simeq H$, iff $G \leq H \leq G$.
- The **core of graph** is the minimal graph (in number of vertices) in equivalency class of $\simeq$
- The **homomorphism order** is partial order induced by $\leq$ on the class of all isomorphism types of cores.

Universality of the homomorphism order

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  - the class of all finite planar cubic graphs
  - the class of all connected series parallel graphs of girth $\geq l$
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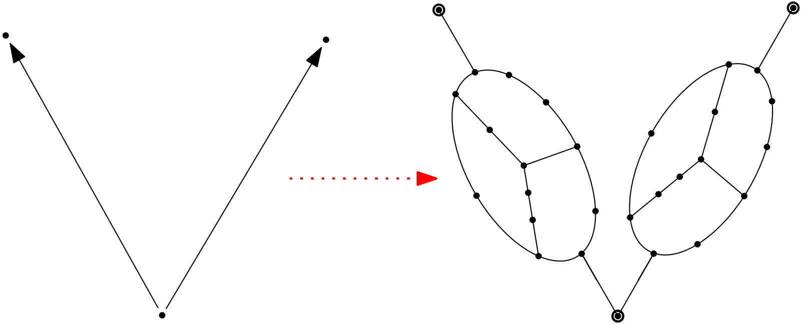
- Dichotomy results on classes of graphs specified by chromatic and achromatic numbers (Nešetřil, Nigussie, 2007)

The arrow (indicator) construction

Main tool: start with oriented paths and transform it to new class by arrow construction



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- Nešetřil 1971: Every locally injective homomorphism $f : G \to G$ is an automorphism of G $\implies$ The **locally injective homomorphism order** is partial order induced by $\leq_i$ on the class of all isomorphism types of connected graphs.

Locally injective homomorphism order

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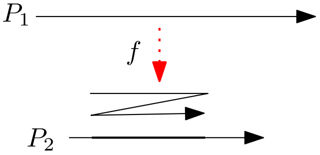
Locally injective homomorphisms are different

Nontrivial homomorphisms of oriented paths involve folding.



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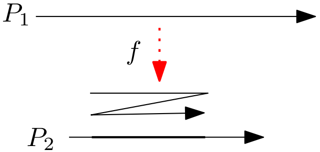
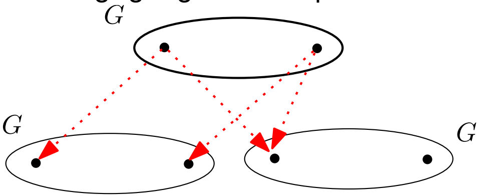


Locally injective homomorphisms never fold
$\Longrightarrow$ need for "folding" gadget G to replace vertices.



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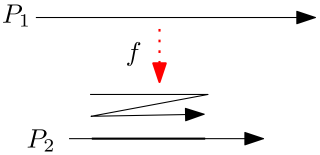

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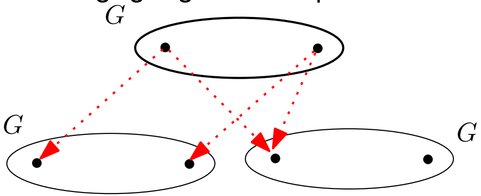

Nešetřil 1971: Every locally injective homomorphism $f : G \to G$ is an automorphism of G.

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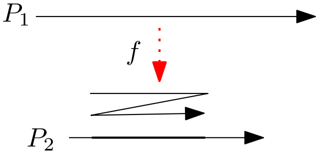

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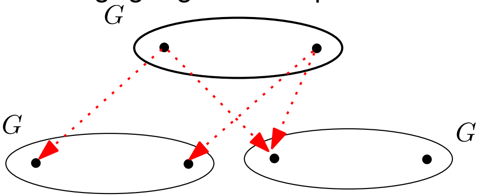

Nešetřil 1971: Every locally injective homomorphism $f : G \to G$ is an automorphism of G. $\Longrightarrow$ no "folding" gadget.

Revisiting argument for universality

- $\mathbb{P}$ is any countably infinite set.
- $P_f(\mathbb{P})$ is a class of finite subsets of $\mathbb{P}$.
- Well known: Every finite partial order $(A, \leq_A)$ can be represented as a suborder of $(P_f(\mathbb{P}), \subseteq)$.
  - Assign $a \in A$ unique $p(a) \in \mathbb{P}$.
  - Represent $a \in A$ by $\{p(b) | b \in A, b \leq_A a\}$.

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  - Assign $a \in A$ unique $p(a) \in \mathbb{P}$.
  - Represent $a \in A$ by $\{p(b) | b \in A, b \leq_A a\}$.
- Partial order is **past-finite** if every down-set is finite.

Lemma

Every past-finite partial order can be represented as a suborder of $(P_f(\mathbb{P}), \subseteq)$*.*

or

$(P_f(\mathbb{P}), \subseteq)$ *is* **past-finite-universal***.*

Cycles are past-finite-universal

- $\mathbb{P}$ is now **set of all odd primes.**
- $A, B \in P_f(\mathbb{P})$ we have $A \subseteq B$ iff

$$\prod_{b \in B} b \text{ is divisible by } \prod_{a \in A} a.$$

Lemma

The divisibility order is past-finite-universal.

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- C_l is oriented cycle of length l.
- $C_l \leq_i C_k$ iff l is divisible by k.

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- C_l is oriented cycle of length l.
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Partial order is **future-finite** if every up-set is finite.

Lemma

Locally injective homomorphism order on the class of all cycles is future-finite-universal.

Future-finite-universal to universal

For partial order $(F, \leq_F)$ we denote by $(P_f(F), \leq_F^{\mathrm{dom}})$ the **subset partial order** where

$$A \leq_F^{\mathrm{dom}} B \iff \text{for every } a \in A \text{ there exists } b \in B \text{ such that } a \leq_F b.$$

Lemma

If $(F, \leq_F)$ is future-finite-universal then $(P_f(F), \leq_F^{\mathrm{dom}})$ is universal.

Future-finite-universal to universal

$A \leq_P^{\mathrm{dom}} B \iff$ for every $a \in A$ there exists $b \in B$ such that $a \leq_P b$.

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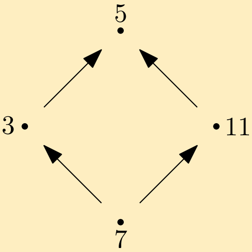

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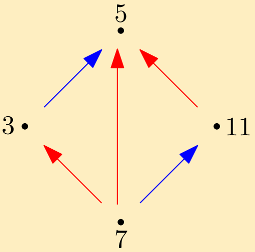

- Partial order $(P, \leq_P)$ with $P \subseteq \mathbb{N}$
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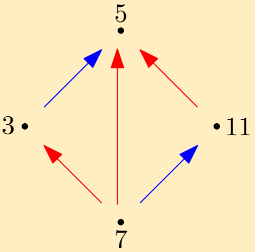

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- Embedding $F : (P, \leq_b) \to (F, \leq_F)$

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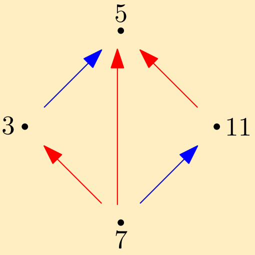

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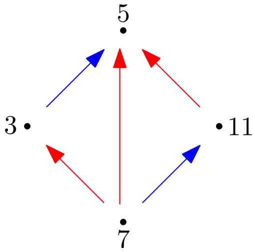
- Embedding $E : (P, \leq_P) \to (P_f(F), \leq_F^{\mathrm{dom}})$
  For $p \in P$ put $E(p) = \{F(p) | p \in P, p \leq_f p\}$.

Putting things together



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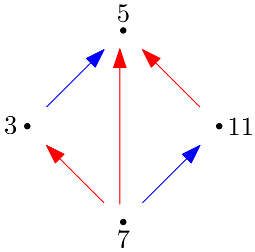

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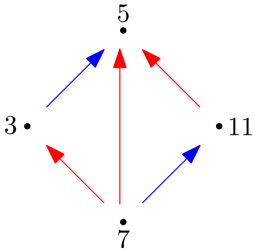

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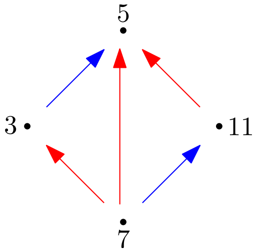

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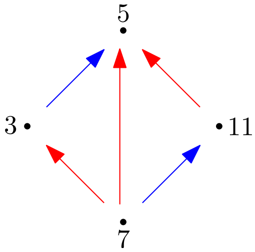

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Locally injective homomorphisms are universal on the class of disjoint unions of cycles.

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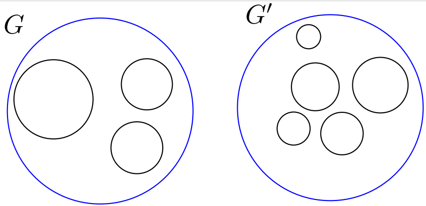

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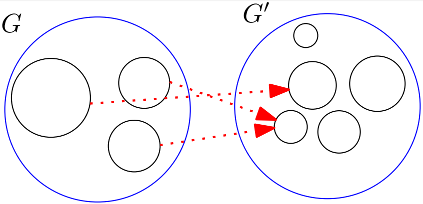

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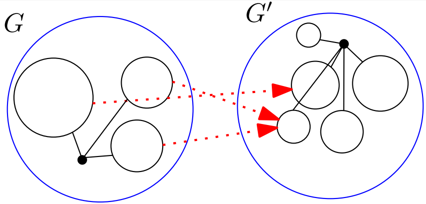

Can not introduce universal vertex to connect components.

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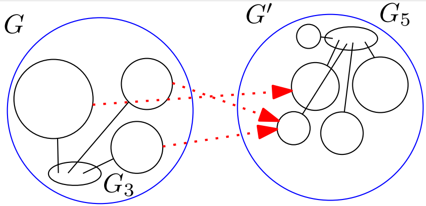
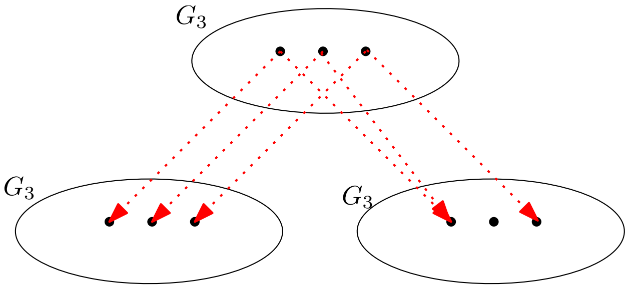
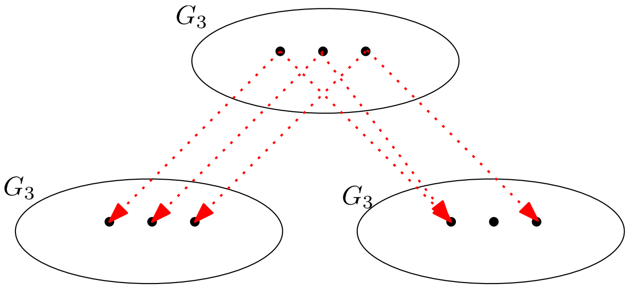


Need connecting gadget G_n for n cycles.

Connecting gadget



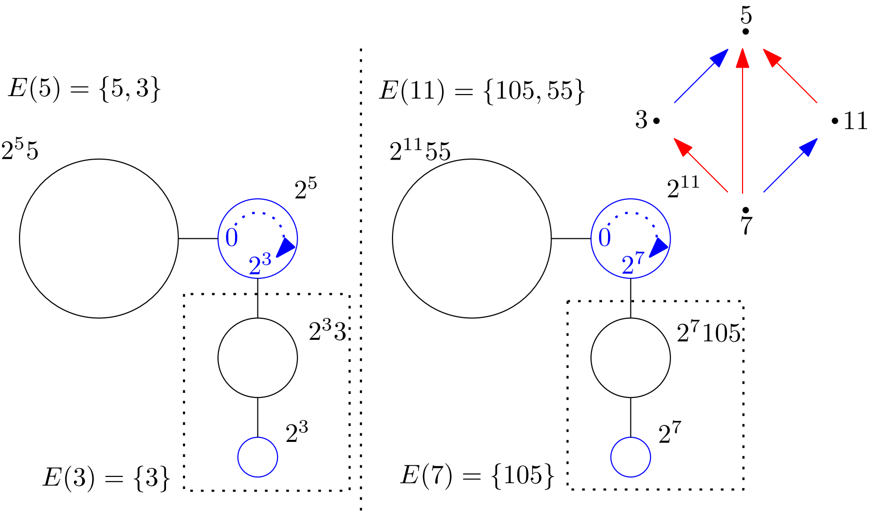
Connecting gadget



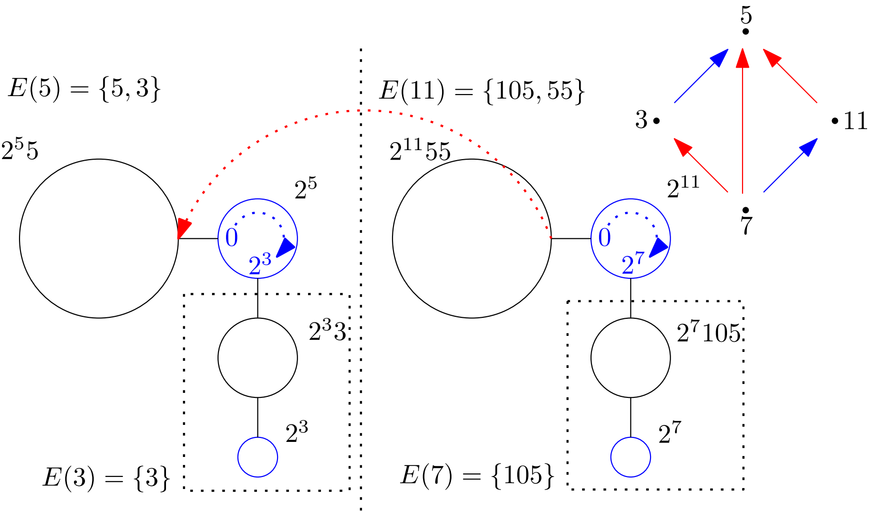
Nešetřil 1971: Every locally injective homomorphism $f : G \to G$ is an automorphism of G

$\implies$ no universal connecting gadget

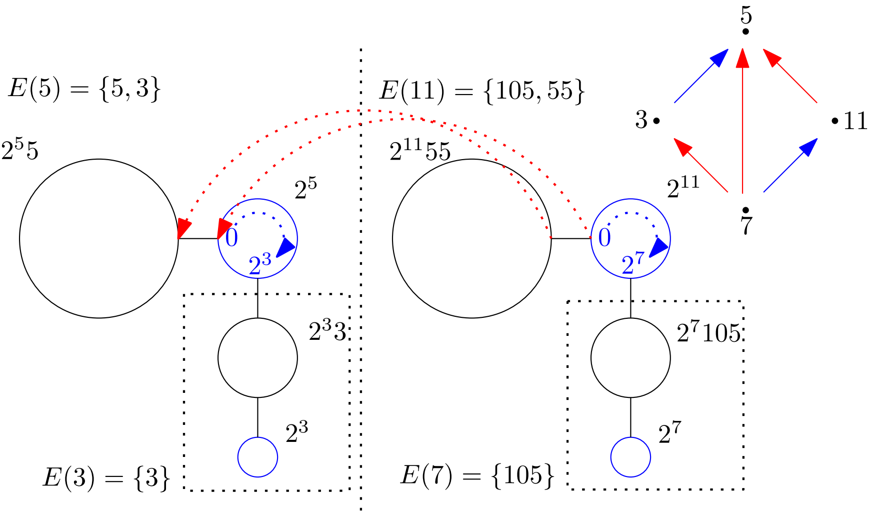
Fractal like structure



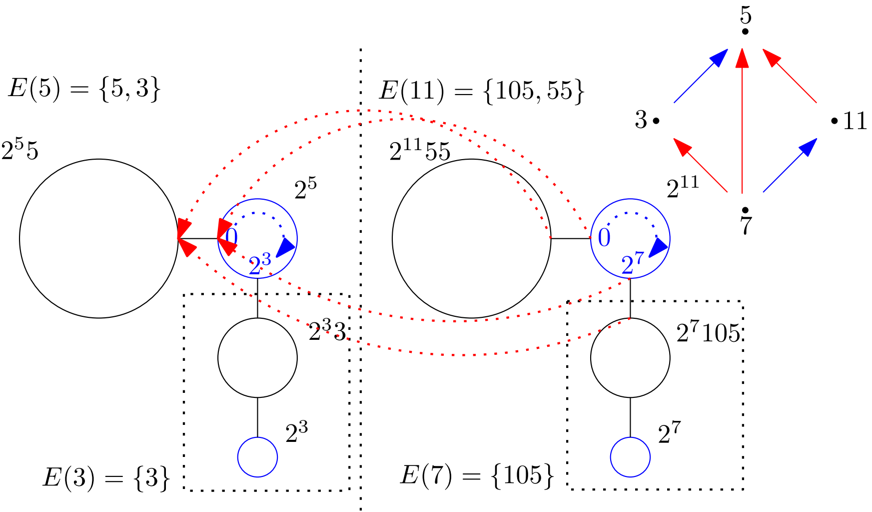
Fractal like structure



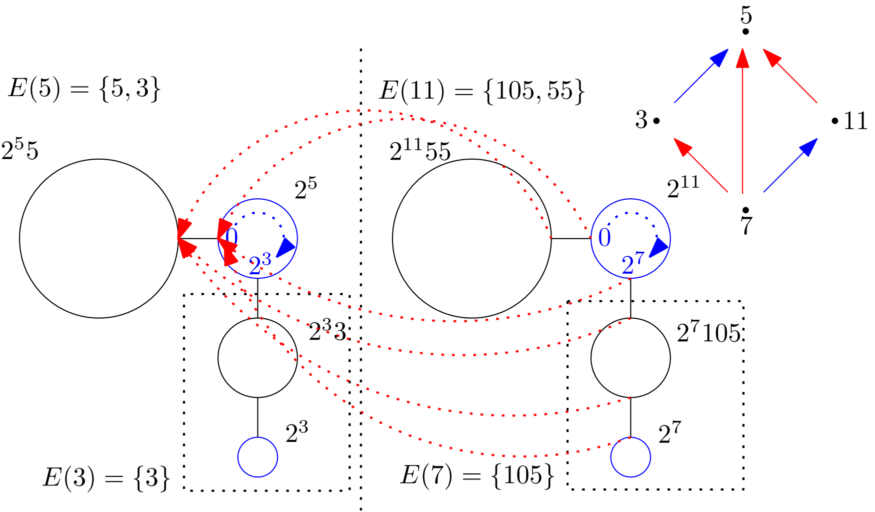
Fractal like structure



Fractal like structure



Fractal like structure



Thank you. . .