Type-respecting amalgamation and big Ramsey degrees

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Joint work with Andres Aranda, Samuel Braunfeld, David Chodounský, Matěj Konečný, Jaroslav Nešetřil, Andy Zucker

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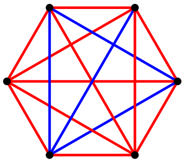
Ramsey theorem and Partition arrow

Theorem (Finite Ramsey Theorem, 1930)

$$\forall_{n,p,k\geq 1}\exists_N : N \longrightarrow (n)^p_{k,1}.$$

$N \longrightarrow (n)^p_{k,t}$: For every partition of $\binom{N}{p}$ into k classes (colours) there exists $X \in \binom{N}{n}$ such that $\binom{X}{p}$ belongs to at most t parts.

($t = 1$ means that $\binom{X}{p}$ is monochromatic.)



For $p = 2$, $n = 3$, $k = 2$ put $N = 6$

Structural Ramsey theorem

Let L be a purely relational language with binary relation $\leq$.

Denote by $\overrightarrow{Rel}(L)$ the class of all finite L-structures where $\leq$ is a linear order of vertices.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)}\exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}.$$

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings $\mathbf{A} \to \mathbf{B}$.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-colouring $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $e \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\{e \circ f : f \in \binom{\mathbf{B}}{\mathbf{A}}\}$ belongs to at most t parts.

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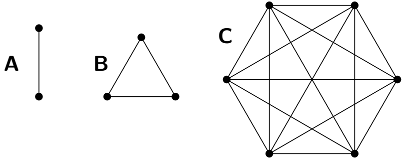

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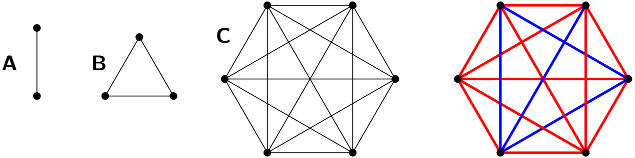

Examples of known Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

1. Class of all finite linear orders
   - Ramsey Theorem, 1930
2. Class of all finite boolean algebras
   - Graham–Rothschild, 1971
3. Class of all finite ordered L-structures, for relational language L
   - Nešetřil–Rödl, 76; Abramson–Harrington, 78
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Area was revitalized in 2005 by the Kechris–Pestov–Todorcevic correspondence

1. Class of all ordered metric spaces
   - Nešetřil, 2005
2. Class of all finite ordered L-structures, for L with relations and functions
   - H.–Nešetřil, 2016
3. General sufficient condition, H., Nešetřil 2019

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Sierpiński: not true for $|O| = 2$.

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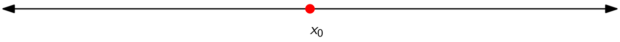

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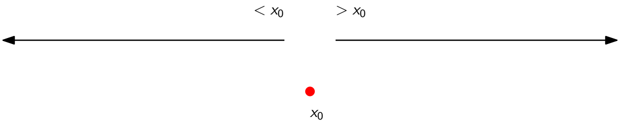

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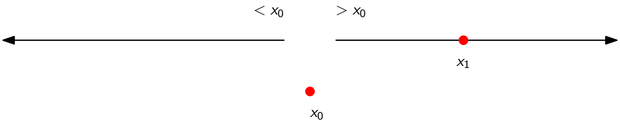

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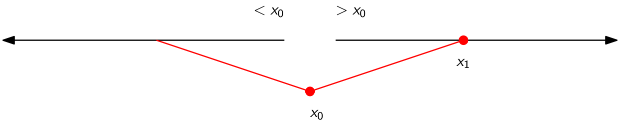

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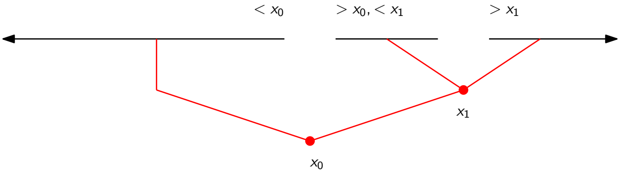

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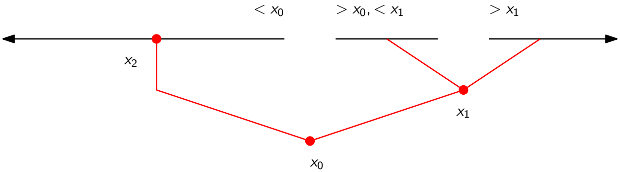

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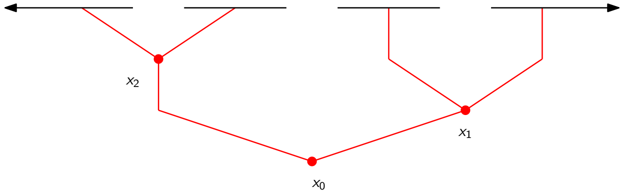

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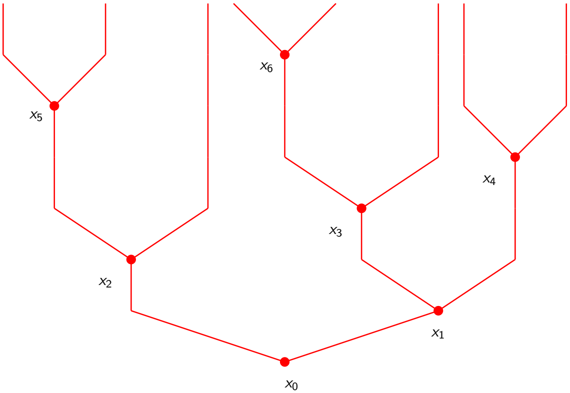

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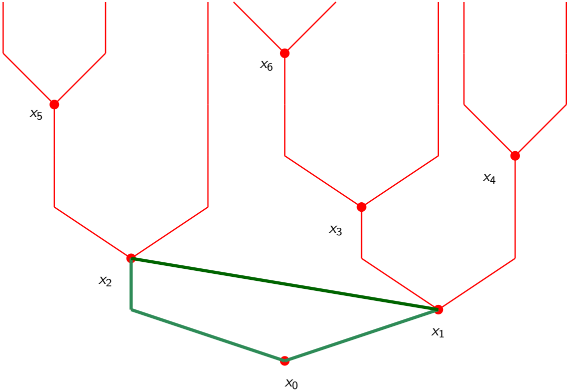

Colour of k-tuple = shape of meet closure in the tree

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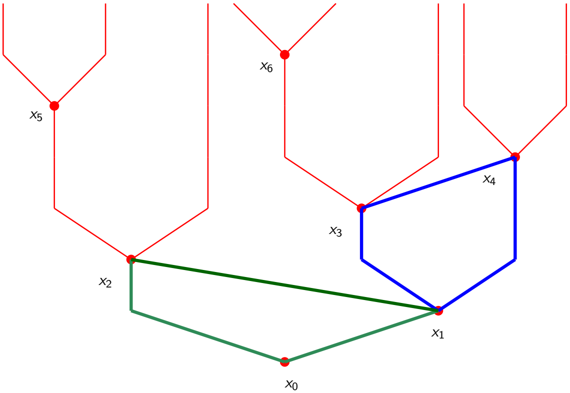

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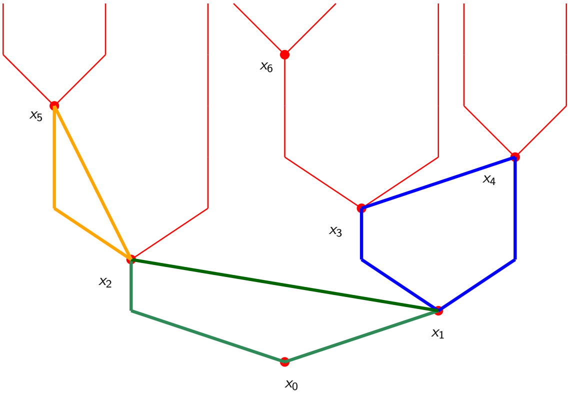

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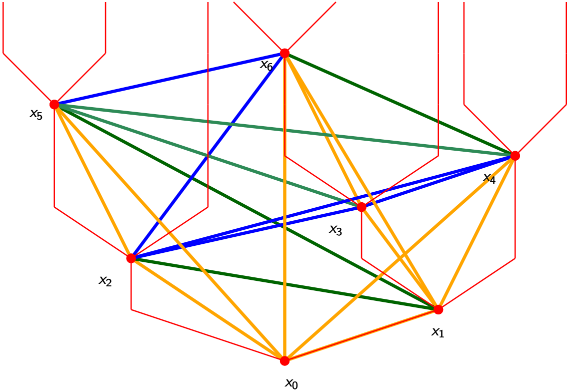

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Big Ramsey Degrees of $(\mathbb{Q}, \leq)$

In late 1960's Laver developed method of finding copies of $\mathbb{Q}$ in $\mathbb{Q}$ with bounded number of colours using Milliken's tree theorem.

Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q}, \leq) \longrightarrow (\mathbb{Q}, \leq)^{(O,\leq_O)}_{k,T}.$$

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$$T(1) = 1, T(2) = 2, T(3) = 16, T(4) = 272,$$

$$T(5) = 7936, T(6) = 353792, T(7) = 22368256$$

Examples of structures with finite big Ramsey degree

Definition (Universal structure)

Let $\mathcal{K}$ be class of finite or countably infinite structures. Structure $\mathbf{A} \in \mathcal{K}$ is universal if every $\mathbf{B} \in \mathcal{K}$ has embedding to $\mathbf{A}$.

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New structures with finite big Ramsey degrees

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6. ω-categorical relational structures
   - Braunfeld, Chodounský, de Rancourt, H., Kawach, Konečný (2023+, 21 pages)

Known big Ramsey results by proof techniques

Ramsey's Theorem

ω, Unary languages

Ultrametric spaces

Λ-ultrametric

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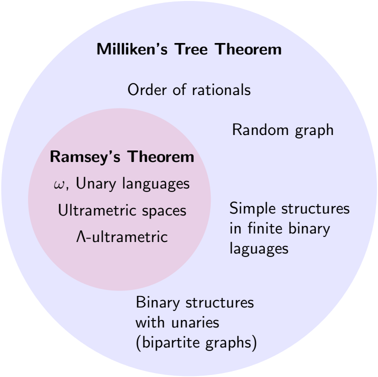

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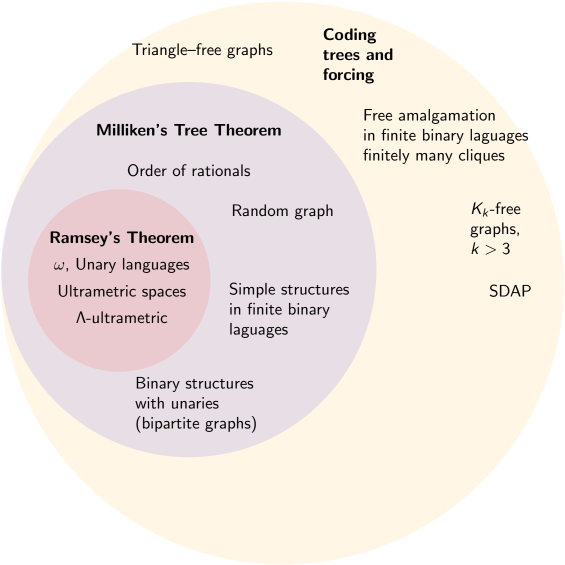

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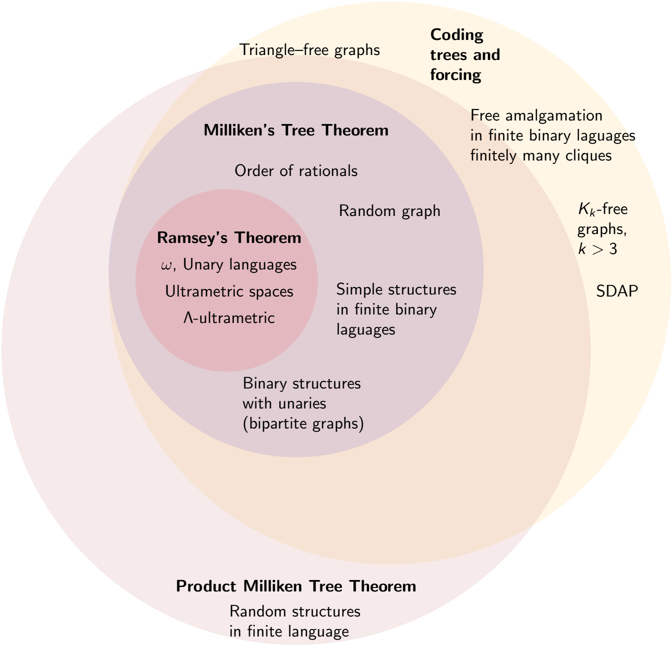

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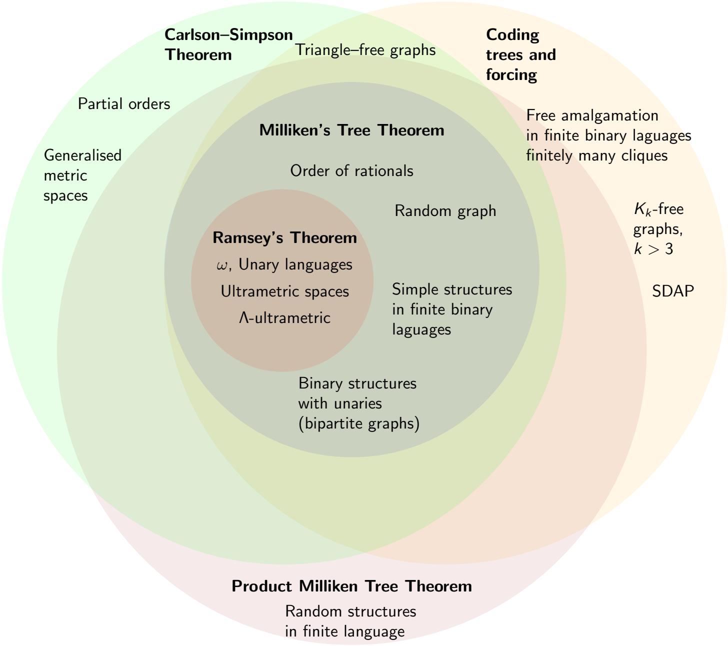

Unified proof technique

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Our main motivation was to generalize

Theorem (Nešetřil-Rödl 1976)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)}\exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}.$$

Moreover **C** *can be constructed such that every irreducible substructure of* **C** *has embedding to* **B**.

L-structure **A** is irreducible if for every pair of vertices u, v of **A** there exists relation symbol $R \in$ L and tuple $\vec{x} \in R_{\mathbf{A}}$ containing both u and v.

Unified proof technique

Balko, Chodounský, Dobrinen, H., Konečný, Nešetřil, Zucker give new abstract Ramsey theorem for trees with sucessor operation which can be used to show all results above (to appear soon).

Our main motivation was to generalize

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Theorem (Zucker 2022)

Let L be a finite relational language with unary and binary relations only and $\mathcal{F}$ a finite family of finite irreducible L-structure. Then the Fraïssé limit of the class of all finite L-structures omitting $\mathcal{F}$ has finite big Ramsey degrees.

Can we drop the additional assumptions?

Generalizing Zucker's theorem

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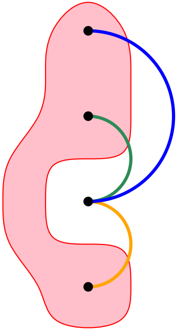
Current proof techniques breaks forbidding



Ordered structures and initial segments

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3. For countable structures the ordering of vertices will always be of order type ω.

Main motivation here is that for a big Ramsey degree of a countable structure to be 1 one needs to have vertices ordered by order-type ω.

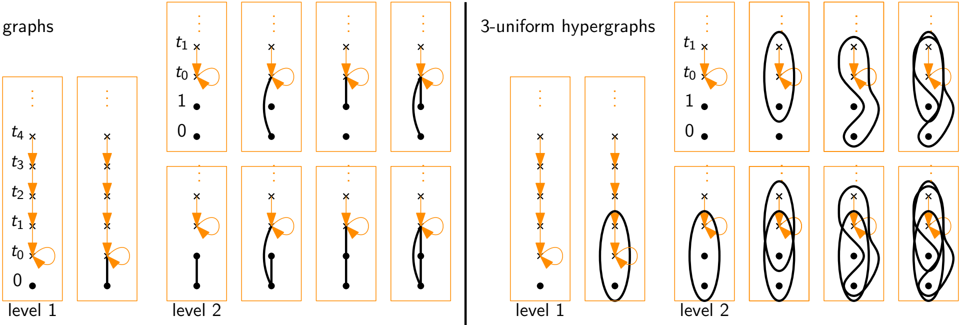
Weak type

We denote by L^f the language L extended by (partial) unary function symbol f.

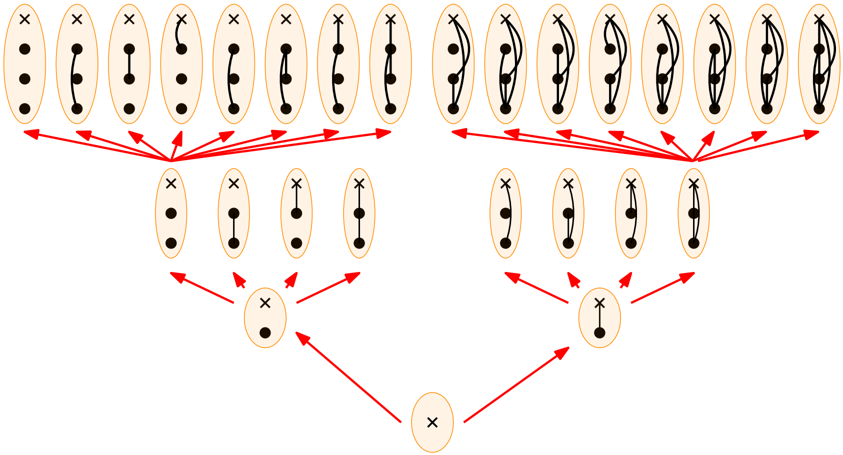
Definition (Weak type)

We denote by L^f the language L extended by unary function symbol f. An L^f-structure **T** is a weak type of level ℓ if

1. $T = \{0, 1, \ldots, \ell - 1, t_0, t_1, \ldots\}$ where vertices t_i are called type vertices.
2. For every $R \in L$ and $\vec{t} \in R_\mathbf{T}$ it holds that $\vec{t} \cap \{t_0, t_1, \ldots\}$ is a initial segment of type vertices and $\vec{t} \cap \{0, 1, \ldots, \ell - 1\} \neq \emptyset$.
3. For every $i > 0$ we put $F_\mathbf{T}(t_i) = t_{i-1}$, $F_\mathbf{T}(t_0) = t_0$, and $F_\mathbf{T}$ is undefined otherwise.



Tree of weak types



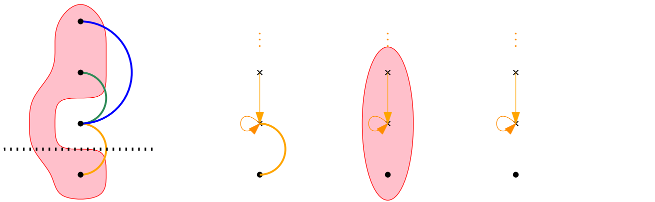
Weak types

Definition (Weak type of a tuple)

Let **A** be an enumerated L-structure, **T** a weak type of level $\ell \in A \subseteq \omega$ and $\vec{a} = (a_0, a_1, \ldots, a_{k-1})$ an increasing tuple of vertices from $A \setminus \ell$. We say that $\vec{a}$ has type **T** on level ℓ if the function $h\colon T \to A$ given by:

$$h(x) = \begin{cases} x & \text{if } x \in \ell, \\ a_i & \text{if } x = t_i \text{ for some } i < k \end{cases}$$

has the property that for every $R \in L$ and $\vec{b}$ a tuple of vertices in $\{0, 1, \ldots, \ell - 1, t_0, t_1, \ldots, t_{k-1}\}$ such that $\vec{b} \cap \{t_0, t_1, \ldots\}$ is an initial segment of type vertices and $\vec{b} \cap \{0, 1, \ldots, \ell - 1\} \neq \emptyset$ it holds that $\vec{b} \in R_{\mathbf{T}} \iff h(\vec{b}) \in R_{\mathbf{A}}$.



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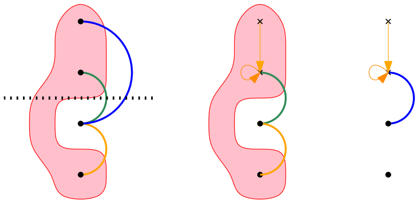


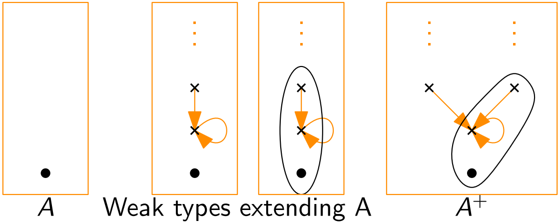
Initial segment with types

1. Given an enumerated L-structure $\mathbf{A}$ and a weak type $\mathbf{T}$, we say that $\mathbf{T}$ extends $\mathbf{A}$ if $\mathbf{T} \setminus \{t_0, t_1, \dots\} = \mathbf{A}$.
2. Given two types $\mathbf{T}$ and $\mathbf{T}'$ that extend $\mathbf{A}$, and $n \geq 0$, we say that $\mathbf{T}$ and $\mathbf{T}'$ agree as n-types if $\mathbf{T} \restriction (A \cup \{t_0, t_1, \dots t_{n-1}\}) = \mathbf{T}' \restriction (A \cup \{t_0, t_1, \dots t_{n-1}\})$.

Definition (Structure with types)

Given a finite enumerated L-structure $\mathbf{A}$, $\mathbf{A}^+$ denotes the L-structure created from the disjoint union of all weak types extending $\mathbf{A}$ by

1. identifying all copies of $\mathbf{A}$, and,
2. identifying the copy of vertex t_i of weak type $\mathbf{T}$ and with the copy of t_i of weak type $\mathbf{T}'$ whenever $\mathbf{T}$ and $\mathbf{T}'$ agree as $i + 1$ types.



Category of well-embeddings

Given an L-structure $\mathbf{A}$ and a vertex v, we denote by $\mathbf{A}(<v)$ the L-structure induced by $\mathbf{A}$ on $\{a \in A; a < v\}$ and call it the initial segment of $\mathbf{A}$.

Definition (Type-respecting embeddings of L-structures)

Given enumerated L-structures $\mathbf{A}$ and $\mathbf{B}$ and an embedding $h\colon \mathbf{A} \to \mathbf{B}$, we say that h is type-respecting if for every $v \in A$ there exists an embedding $h^v : \mathbf{A}(<v)^+ \to \mathbf{B}(<h(v))^+$ such that the weak types of tuples in $\mathbf{B}$ on level $h(v)$ consisting only of vertices of $h[A]$ are all in the image $h^v[\mathbf{A}]$.

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Definition ($\mathcal{K}$-type-respecting embeddings of initial segments)

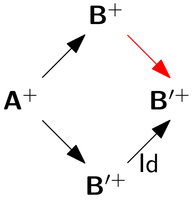
Let $\mathbf{A}$ and $\mathbf{B}$ be two finite enumerated L-structures. Embedding $h\colon \mathbf{A}^+ \to \mathbf{B}^+$ is type-respecting if for every (possibly infinite) L-structure $\mathbf{A}'$ with initial segment $\mathbf{A}$ there exists an L-structure $\mathbf{B}'$ with initial segment $\mathbf{B}$ and a type-respecting embedding $g\colon \mathbf{A} \to \mathbf{B}$ finitely approximated by h. That is $g \restriction A = h \restriction A$ and every weak type in $\mathbf{B}'$ of a tuple consisting of vertices of $g[A]$ of level $g(\max A)$ is in $h[A^+]$.

Given class $\mathcal{K}$ of L-structures we say that $h\colon \mathbf{A}^+ \to \mathbf{B}'^+$ is $\mathcal{K}$-type-respecting if for every L-structure $\mathbf{A}' \in \mathcal{K}$ with initial segment $\mathbf{A}$ there exists an structure $\mathbf{B}' \in \mathcal{K}$ with initial segment $\mathbf{B}$ and a type-respecting embedding $g\colon \mathbf{A} \to \mathbf{B}$ finitely approximated by h.

Amalgamation with types

Definition (Type-respecing amalgamation property)

Let $\mathcal{K}$ be a hereditary class of enumerated L-structures. We say that $\mathcal{K}$ has type-respecting amalgamation property if given three finite enumerated L-structures $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}' \in \mathcal{K}$ such that $B' \setminus B = \{\max B'\}$ and $\mathbf{B}' \restriction \mathbf{B} = \mathbf{B}$, two $\mathcal{K}$-type-respecting embeddings $f\colon \mathbf{A}^+ \to \mathbf{B}^+$, $f'\colon \mathbf{A}^+ \to \mathbf{B}'^+$ and a type-respecting (but not necessarily $\mathcal{K}$-type-respecting) embedding $g\colon \mathbf{B}^+ \to \mathbf{B}'^+$ such that $g \restriction B$ is the identity and $g \circ f = f'$, there exists a $\mathcal{K}$-type-respecting embedding $g'\colon \mathbf{B}^+ \to \mathbf{B}'^+$ such that $g' \circ f = f'$ and $g' \restriction B = \mathrm{Id}$.



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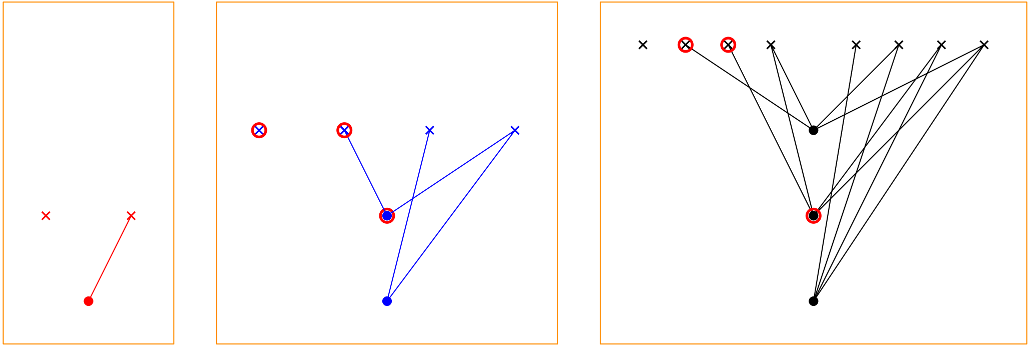

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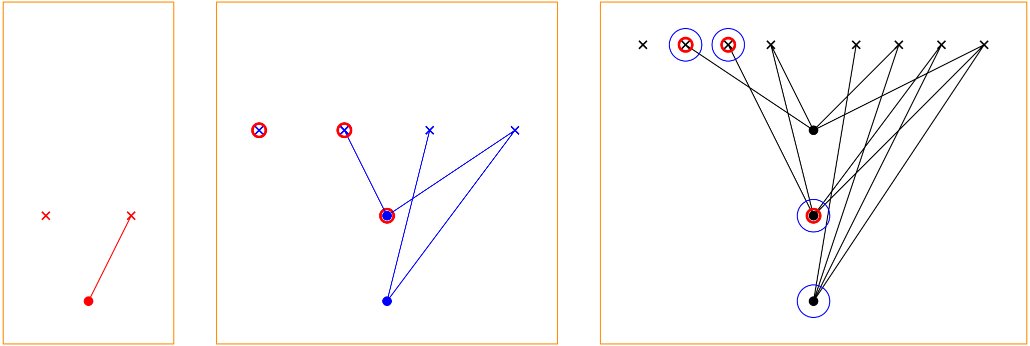

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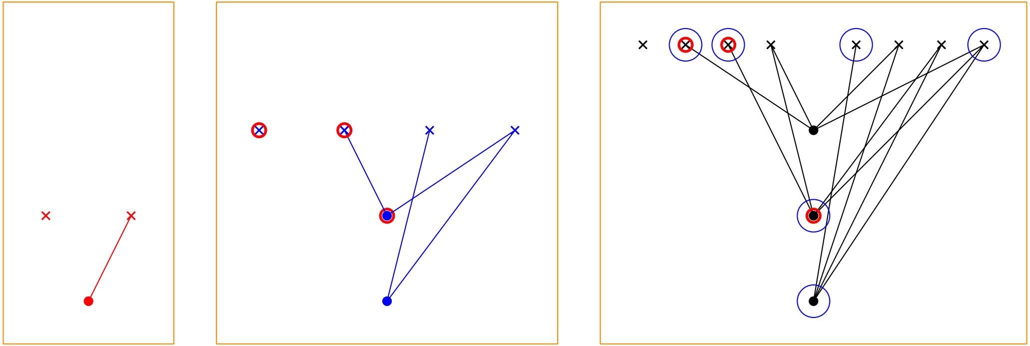


$\mathbf{A}^+$ $\mathbf{B}^+$ $\mathbf{B}'^+$

Big Ramsey degrees for type-resepcting-embeddings

Theorem (Braunfeld, Chodounský, H., Konečný, Nešetřil, Zucker)

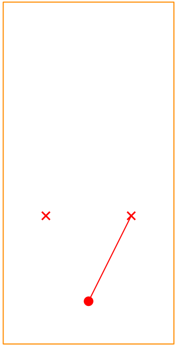
Let L be a finite relational language. Let $\mathcal{F}$ be a finite family of finite irreducible enumerated L-structures. Denote by $\mathcal{K}_\mathcal{F}$ the class of all finite or countably-infinite enumerated L-structures **A** *where $\leq_\mathbf{A}$ is either finite or of order-type ω such that for every* $\mathbf{F} \in \mathcal{F}$ *there no embedding* $\mathbf{F} \to \mathbf{A}$. *Assume that $\mathcal{K}_\mathcal{F}$ has the type-respecting amalgamation property. Then for every universal L-structure* $\mathbf{U} \in \mathcal{K}_\mathcal{F}$ *and every finite* $\mathbf{A} \in \mathcal{K}_\mathcal{F}$ *there is a finite $D = D(\mathbf{A})$ such that* $\mathbf{U} \longrightarrow^{\mathcal{K}} (\mathbf{U})^{\mathbf{A}}_{k,D}$ *for every $k \in \mathbb{N}$.*

Examples

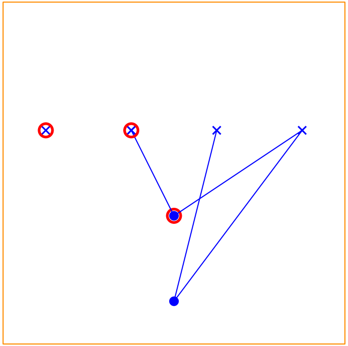
1. All families $\mathcal{F}$ in finite languages consisting of unary and binary languages.
2. Families $\mathcal{F}$ of cliques on 4 vertices in language having ternary symbols such that there is a relation with vertices $1, 2, 3$.

Class of all ordered $\mathcal{F}$-free structures has type-respecing amalgamation property

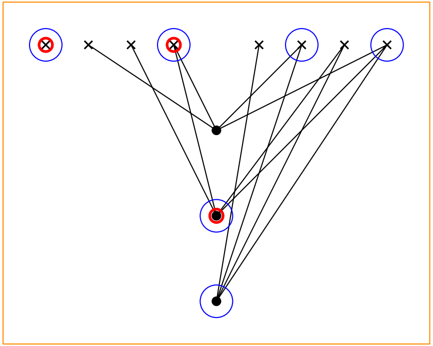
Consider example of triangle free graphs



$\mathbf{A}^+$



$\mathbf{B}^+$

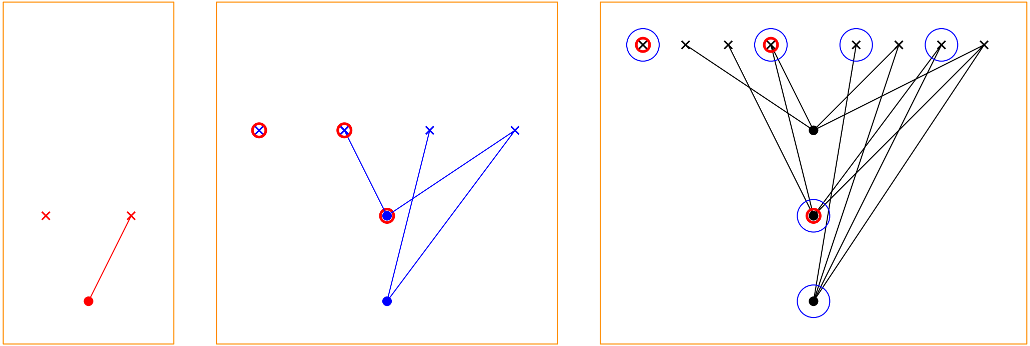


$\mathbf{B}'^+$

Not a typed-amalgamation!

Class of all ordered $\mathcal{F}$-free structures has type-respecing amalgamation property

Consider example of triangle free graphs



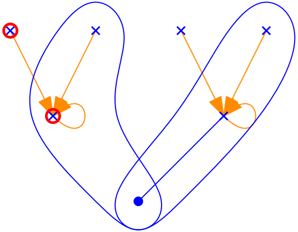
$\mathbf{A}^+$ $\mathbf{B}^+$ $\mathbf{B}'^+$

Typed amalgamation

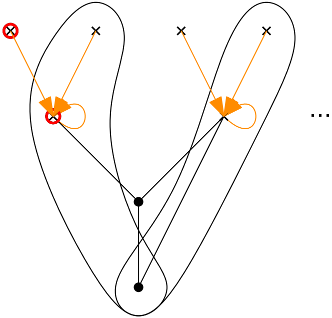
An type-amalgamation failure for free amalgamation class

Forbidden structure

$\mathbf{A}^+$



$\mathbf{B}^+$



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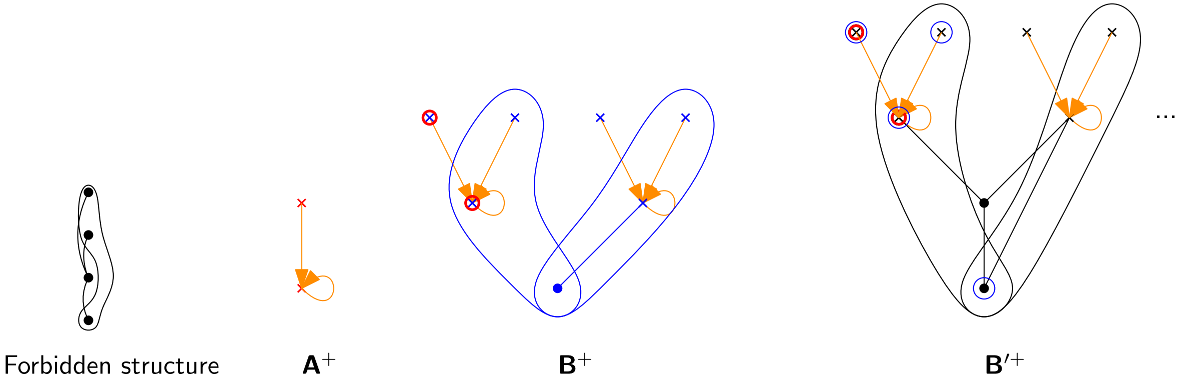
An type-amalgamation failure for free amalgamation class



Work in progress and open problems

1. Generalization to strong amalgamation classes (metric spaces, posets and other structures has typed amalgamation)
2. Can we construct bad colouring for the well-emebedding category if typed amalgamation fails?
   (we have bad colouring for pigeonhole of the tree theorem)
3. Can we construct bad coloring for the emebedding category if typed amalgamation fails?
4. Can we show that small Ramsey degrees for well-embeddings implies big Ramsey degrees?

Thank you for the attention

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