Big Ramsey degrees of the 3-uniform hypergraph

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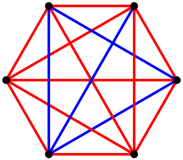
Ramsey Theorem

Theorem (Finite Ramsey Theorem, 1930)

$$\forall_{n,p,k\geq 1}\exists_N : N \longrightarrow (n)^p_{k,1}.$$

$N \longrightarrow (n)^p_{k,t}$: For every partition of $\binom{\{1,2,\ldots,N\}}{p}$ into k classes (colours) there exists $X \subseteq \{1,2,\ldots,N\}$, $|X| = n$ such that $\binom{X}{p}$ belongs to at most t partitions (if $t = 1$ it is monochromatic)



For $p = 2$, $n = 3$, $k = 2$ put $N = 6$

Ramsey theorem for finite structures

Denote by $\overrightarrow{\mathcal{H}}_l$ the class of all finite l-uniform hypergraphs endowed with linear order on vertices.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{l\geq 2, \mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{H}}_l} \exists_{\mathbf{C} \in \overrightarrow{\mathcal{H}}_l} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}.$$

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all induced sub-hypergraphs of **B** isomorphic to **A**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ has at most t colours.

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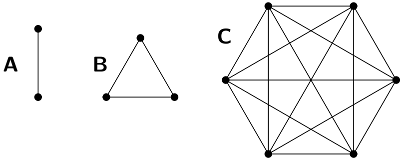

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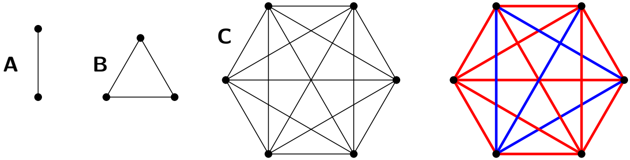
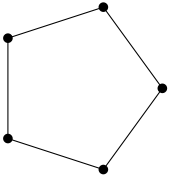


Order is necessary

A

B



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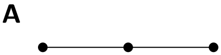
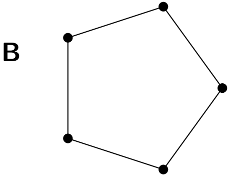
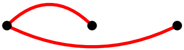




Vertices of **C** can be linearly ordered and copies of **A** colored:

- red if middle vertex appear first.



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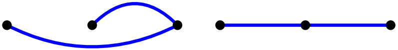

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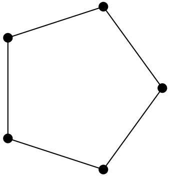

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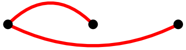

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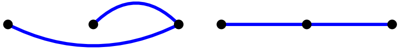
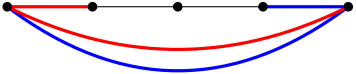


Every ordering of 5-cycle contains minimal and maximal element.
Consequently every 5-cycle in **C** with contain both blue and red copy of **A**.



Hypergraphs have finite small Ramsey degree

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where $t(\mathbf{A})$, the small Ramsey degree of $\mathbf{A}$ in $\mathcal{H}_l$, is the number of non-isomorphic ordering of vertices of $\mathbf{A}$.

Ramsey classes

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A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}$.

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Example (Models — H.-Nešetřil, 2016)

For every language L, $\overrightarrow{Mod}(L)$ is a Ramsey class.

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words

Milliken tree theorem: C-relations

Ramsey's theorem: rationals

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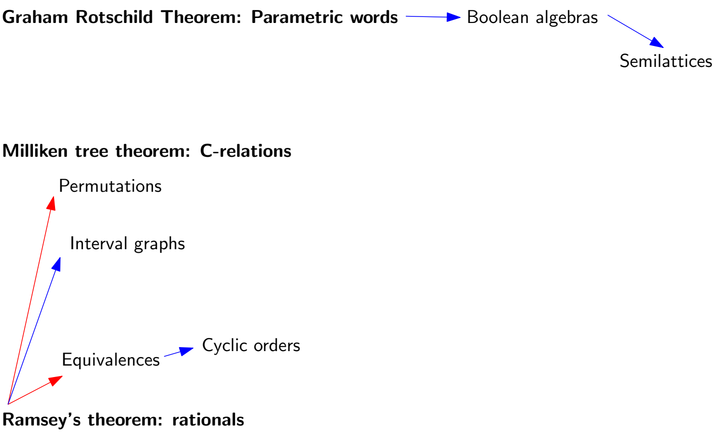
Permutations

Equivalences

Ramsey's theorem: rationals

Product arguments

Gower's Ramsey Theorem



Product arguments   Interpretations

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Graham Rotschild Theorem: Parametric words → Boolean algebras → Semilattices

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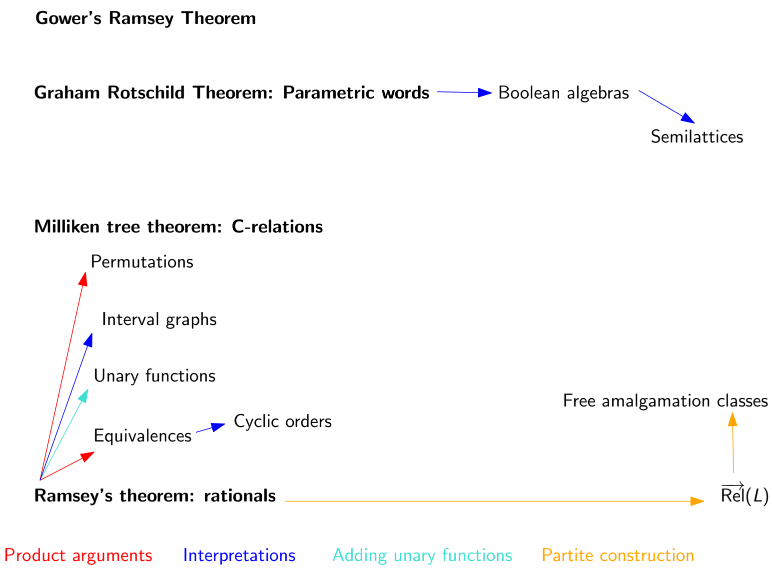
Equivalences → Cyclic orders

Ramsey's theorem: rationals

Product arguments   Interpretations   Adding unary functions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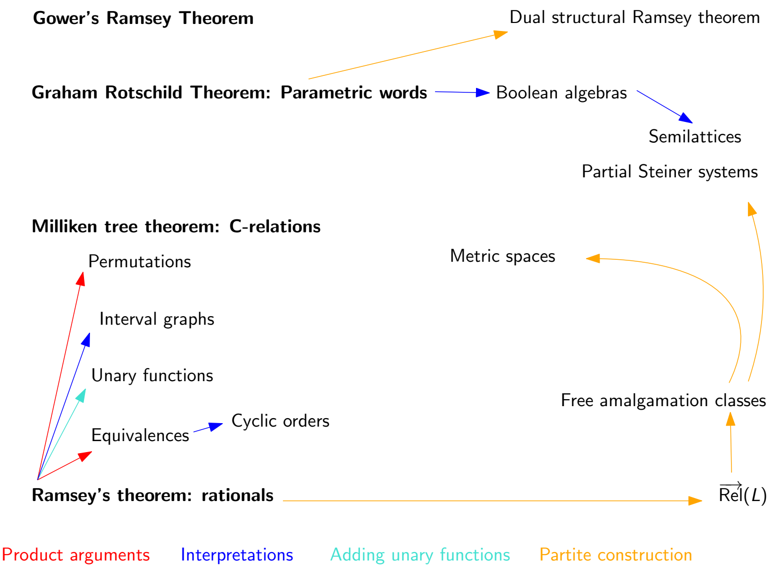
Gower's Ramsey Theorem
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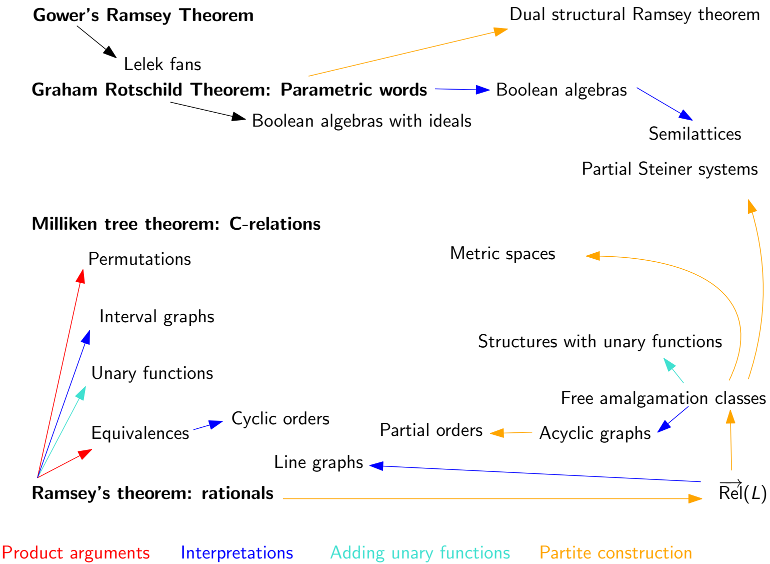
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Ramseyness is an application of Milliken's tree theorem on binary tree.

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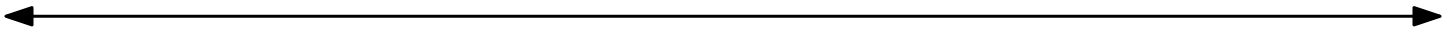

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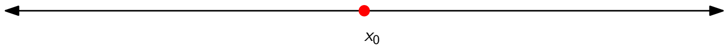

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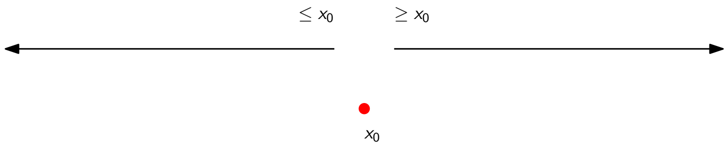

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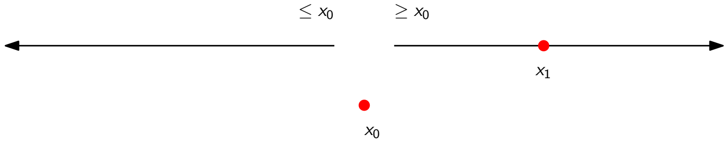

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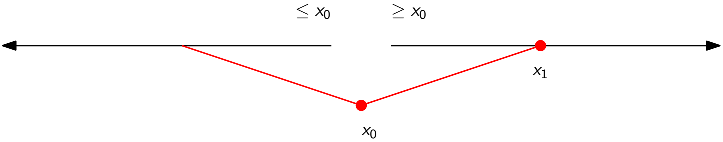

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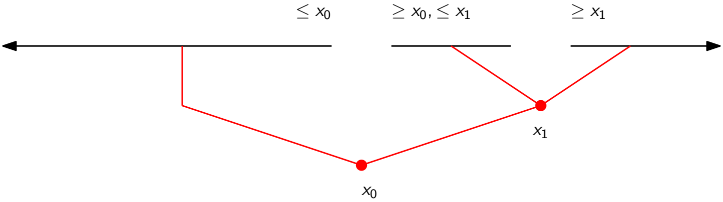

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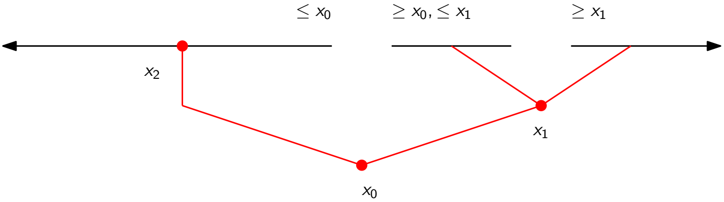

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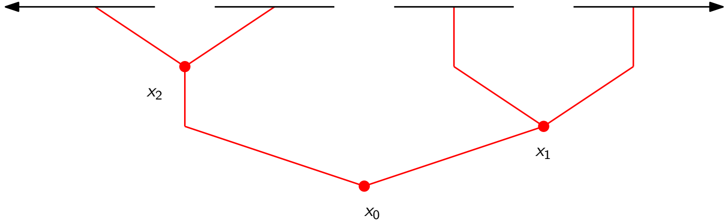

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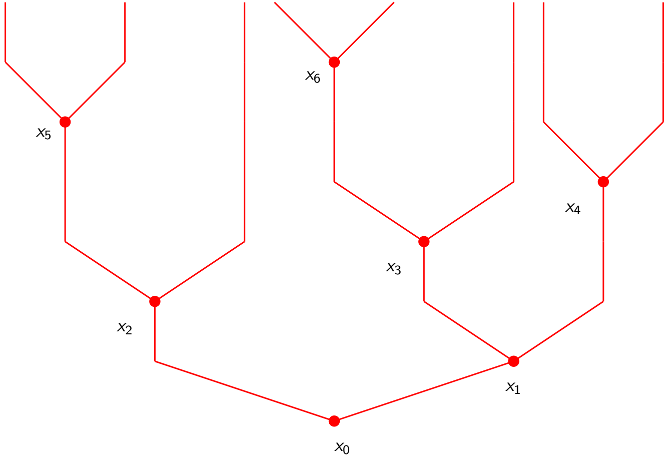

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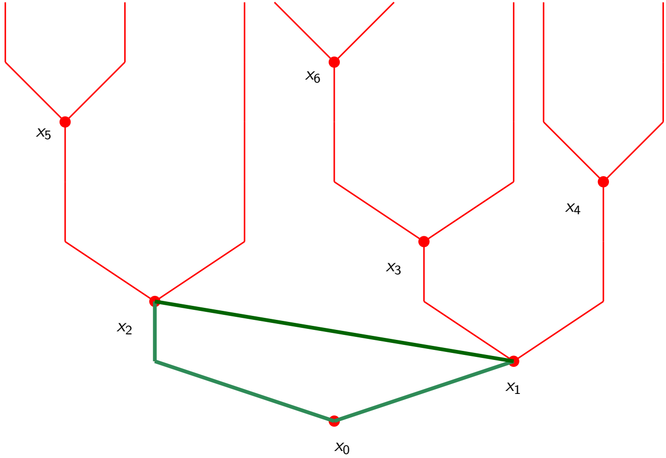

Colour of k-tuple = shape of meet closure in the tree

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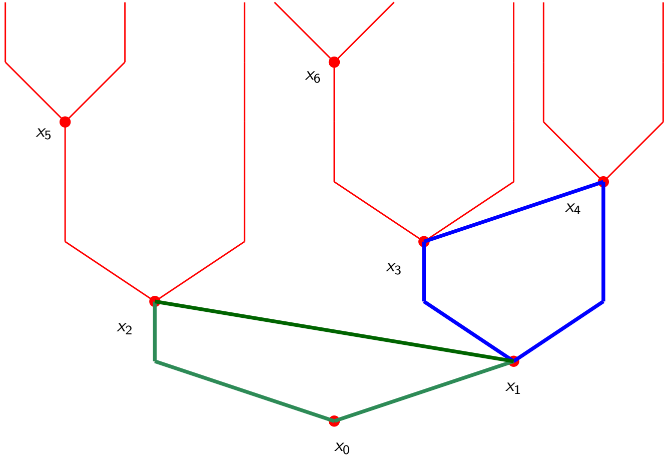

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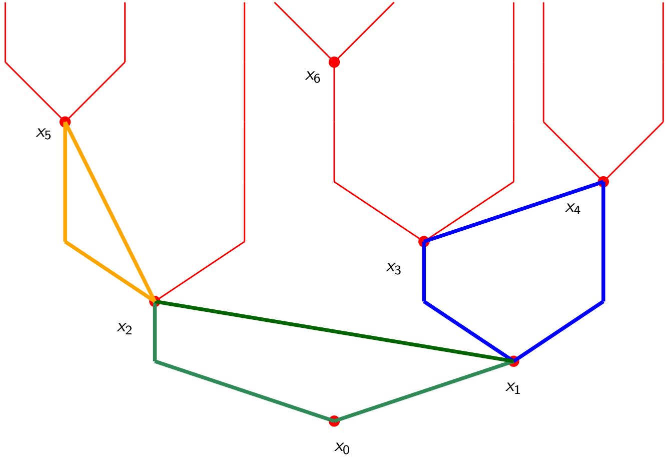

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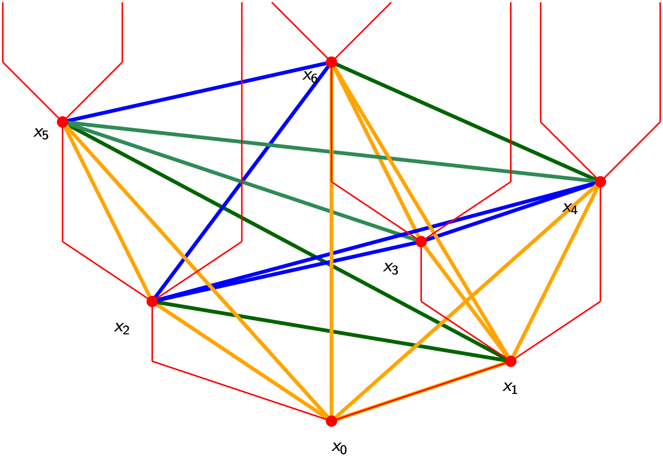

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Big Ramsey degrees of $\mathbb{Q}$ as trees

We describe tree using two orders.

1. $\leq$ is the order of rationals
2. $\preceq$ is the well-order fixed by enumeration

$T(X, \leq, \preceq)$ is the tree built by previous procedure for set $(X, \leq)$ executed in order given by $\preceq$. (A binary search tree used in computer science.)

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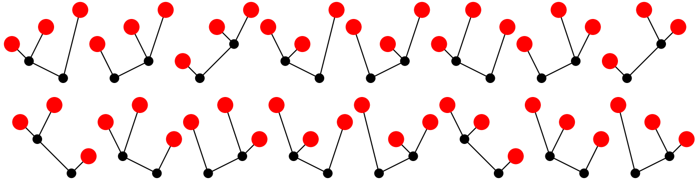

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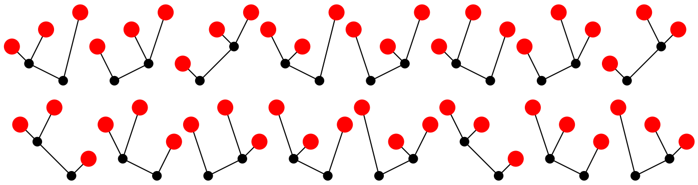

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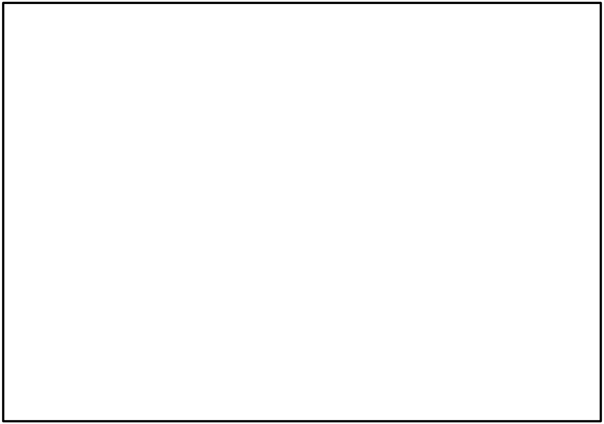
The big Ramsey degree of n tuples is precisely given by number of compatible pair of orders on an n-tuple.

Rich colouring of the countable random graph

R



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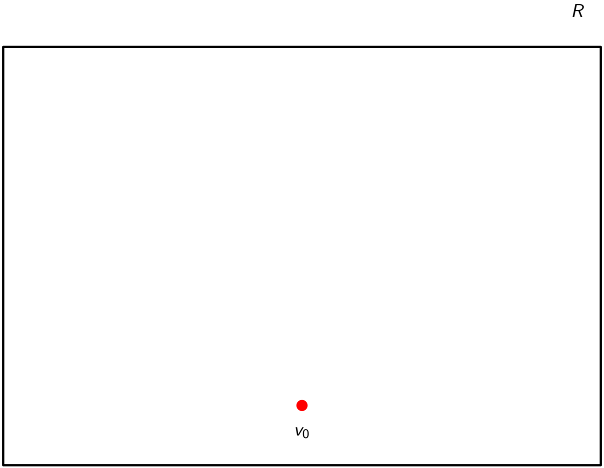
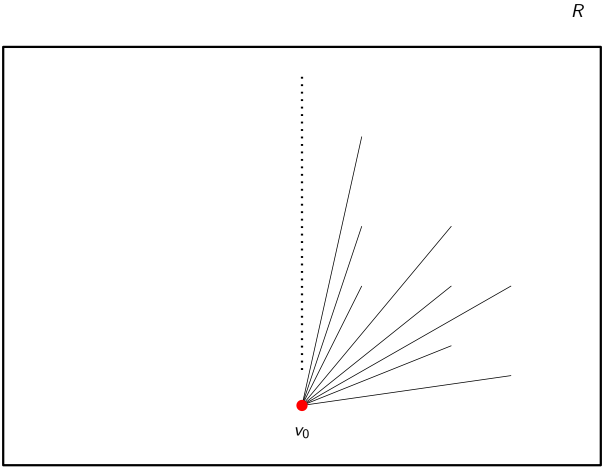


Rich colouring of the countable random graph



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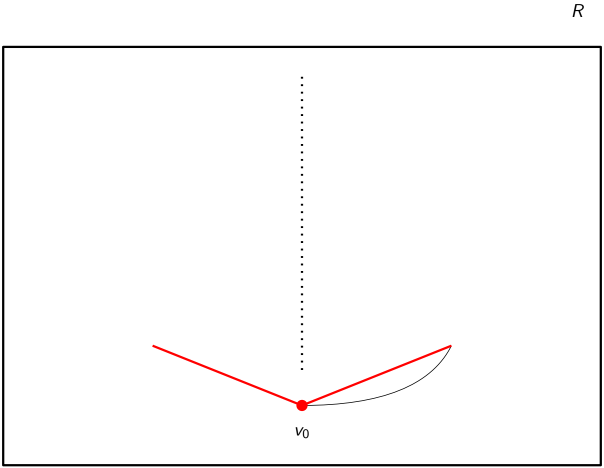

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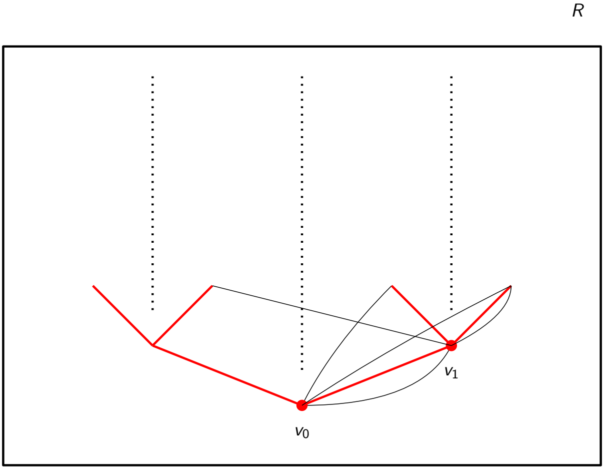

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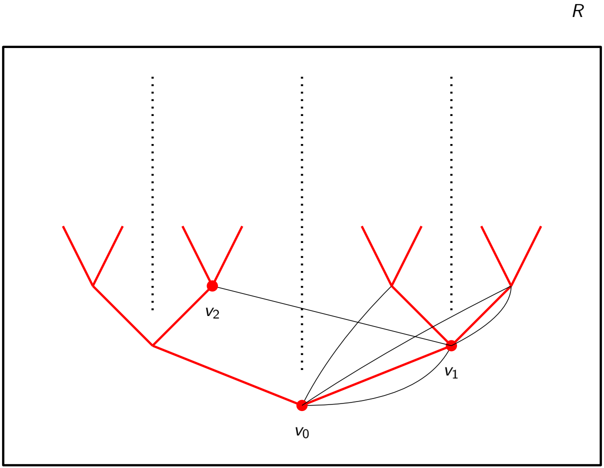

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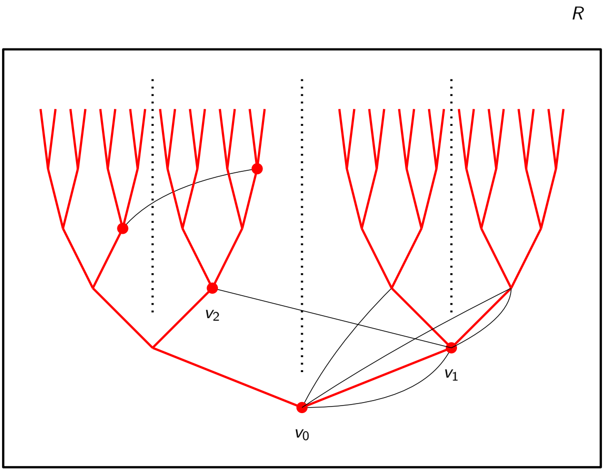

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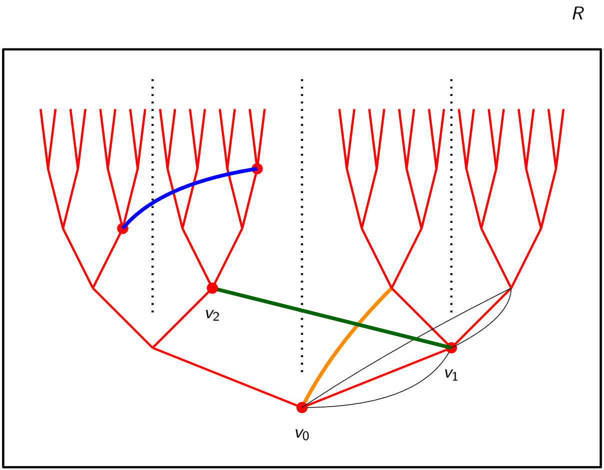
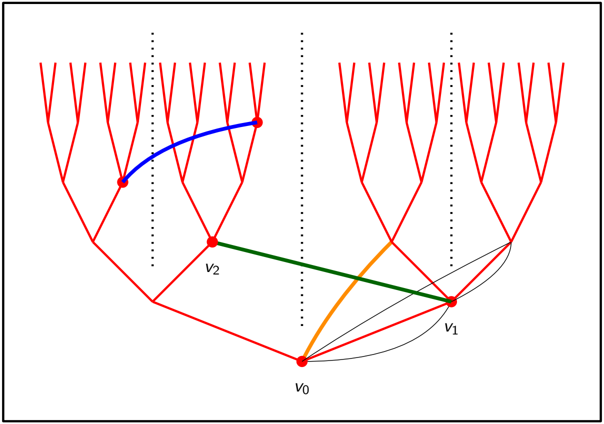


Colour of a subgraph = shape of meet closure in the tree

Rich colouring of the countable random graph

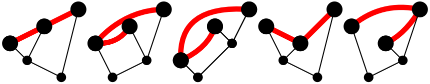
R



Colour of a subgraph = shape of meet closure in the tree
Given well order $\preceq$ (bottom-up) one can define dense order by listing the tree from left to right

Sauer's theorem

Big Ramsey degrees of graphs are given by Devlin's trees annotated by graph edges.



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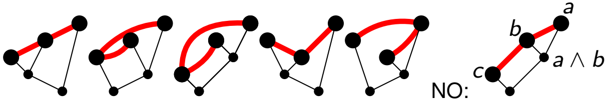

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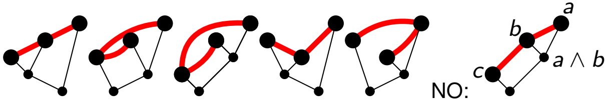

Definition

Let $(\leq, \preceq)$ be a pair of compatible orders of a set V', let V be the set of leaf vertices of $T(V', \leq, \preceq)$, and let $G = (V, E)$ be a graph. We say that G is compatible with $T(V', \leq, \preceq)$ if, for every triple $a, b, c \in V$ of distinct vertices satisfying $c \preceq (a \wedge b)$, we have $\{a, c\} \in E$ if and only if $\{b, c\} \in E$.

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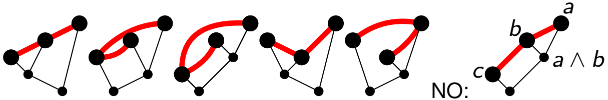

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Theorem (Sauer 2006)

Let R be the random graph.

$$\forall_{\textit{finite graph } Gk \geq 1} R \longrightarrow (R)^{G}_{k, T(H)}.$$

Where $T(G)$ is the Ramsey degree of a graph Ramsey degree of a graph $G = ([n], E)$ in R.

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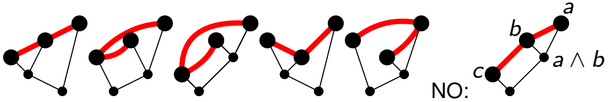

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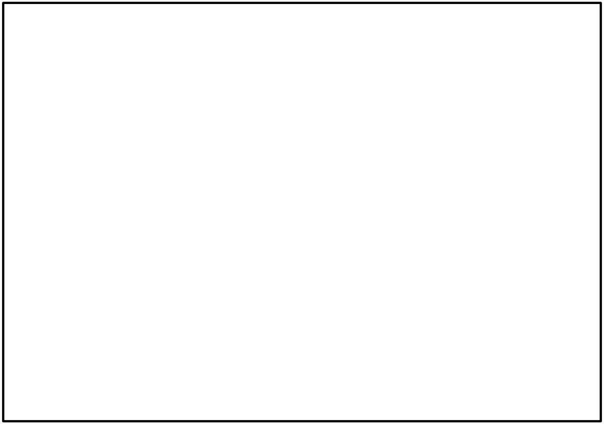
$T(G)$ is the number of non-isomorphic structures $([2n-1], E, \leq, \preceq)$ where $(\leq, \preceq)$ is a pair of compatible linear orders of $[2n-1]$ and G is compatible with $T([2n-1], \leq, \preceq)$.

Proved again by the application of Milliken tree theorem on binary tree.

Rich colouring of the countable random 3-uniform hyper-graph

H_3



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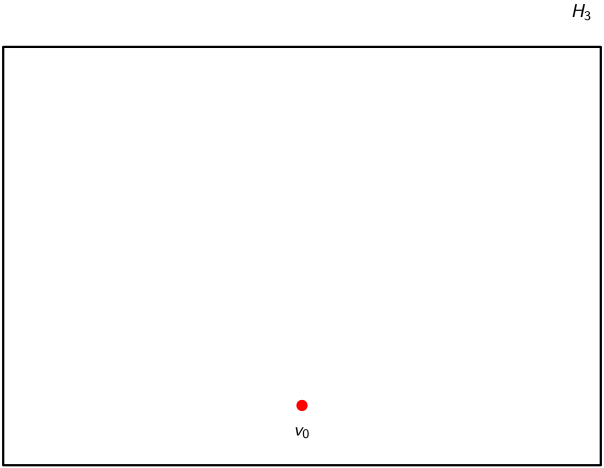

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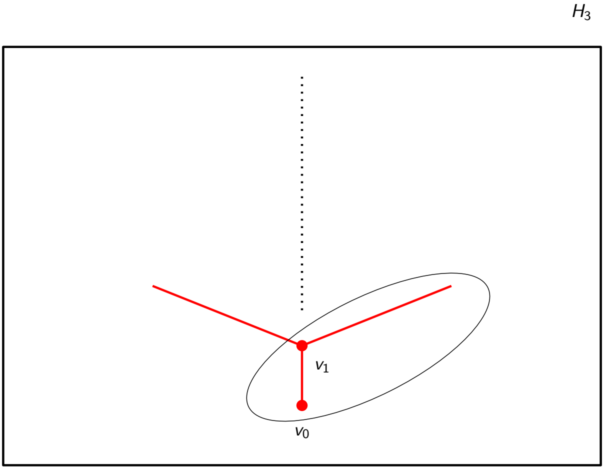

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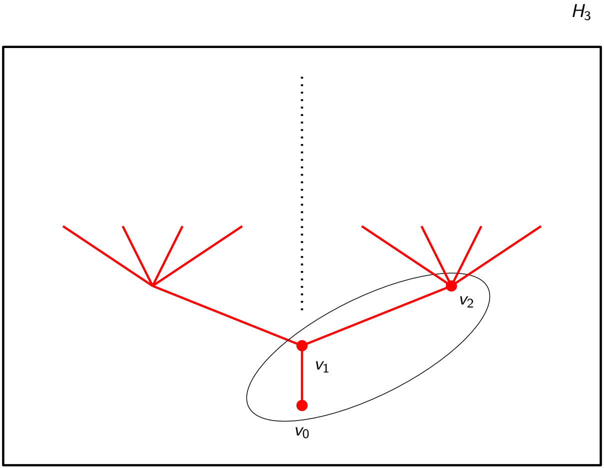

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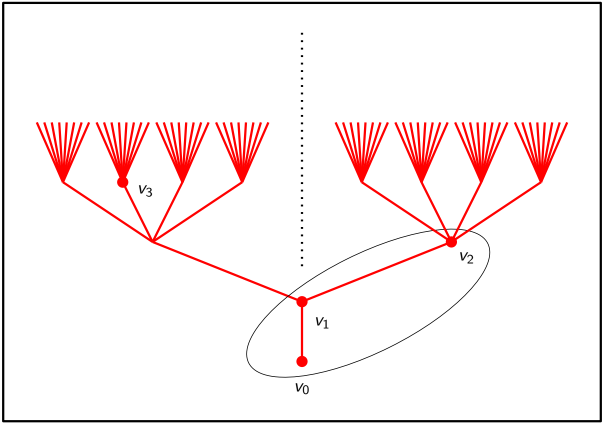
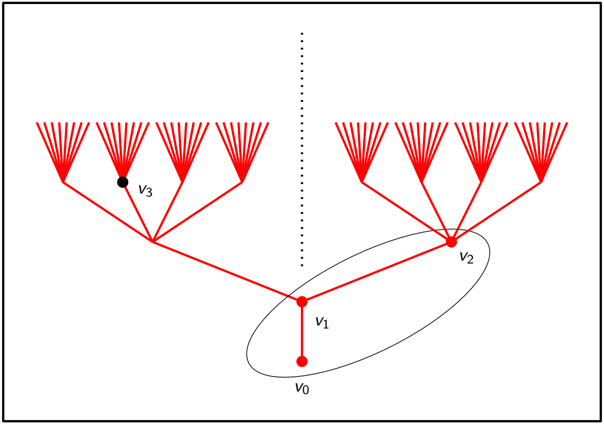


Colour of a subgraph = shape of meet closure in the tree

Problem: Ramsey theorem for this type of tree does not hold

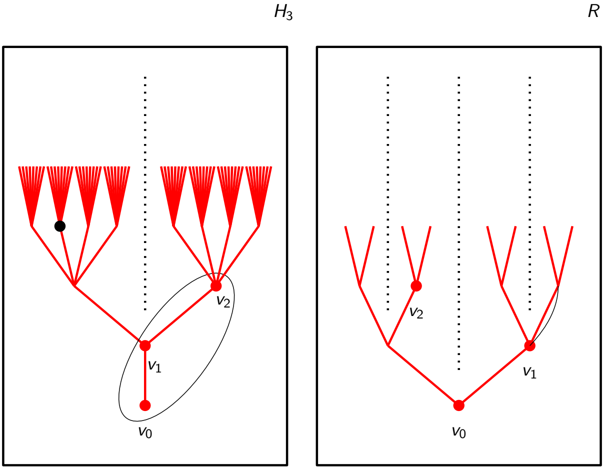
Rich colouring of the countable random 3-uniform hyper-graph

H_3



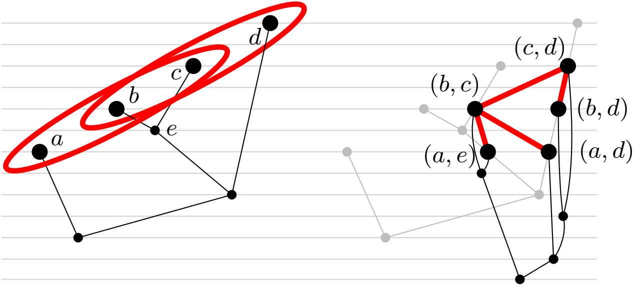
Colour of a subgraph = shape of meet closure in the tree
Problem: Ramsey theorem for this type of tree does not hold
Year later we observed that neighbourhood of a vertex is the Random graph!

Rich colouring of the countable random 3-uniform hyper-graph

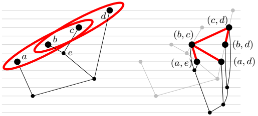


Colour of a subgraph = shape of meet closure in both trees
Problem: Ramsey theorem for this type of tree does not hold
Year later we observed that neighbourhood of a vertex is the Random graph!

Big Ramsey degrees of 3-uniform hypergraphs are pairs of trees



Big Ramsey degrees of 3-uniform hypergraphs as product trees



Definition

Let $(\leq, \preceq)$ be a pair of compatible orders of a set V', let V be the set of leaf vertices of $T(V', \leq, \preceq)$, and let $\mathcal{G} = (V, \mathcal{E})$ be a 3-uniform hypergraph. We say that $\mathcal{G}$ is compatible with $T(V', \leq, \preceq)$ if for every 4-tuple a, b, c, d of distinct vertices of V satisfying $d \preceq c \preceq (a \wedge b)$ we have $\{a, c, d\} \in \mathcal{E}$ if and only if $\{b, c, d\} \in \mathcal{E}$.

Given a tree $T(V^0, \leq, \preceq)$ and a compatible 3-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$, we define the *neighbourhood graph* of $\mathcal{G}$ with respect to $T(V^0, \leq, \preceq)$ as the graph $G^1 = (V'', E^1)$ constructed as follows:

- V'' consists of all pairs (a, b) such that $a \in V$ (by compatibility $V \subseteq V^0$) and $b \in V^0$, $a \prec b$ and there is no $c \in V^0$, $c \sqsubset b$ such that $a \prec c \prec b$.
- $\{(a, b), (c, d)\} \in E^1$ for $a \preceq c$ iff there exists $e \sqsupseteq d$ such that $\{a, c, e\} \in \mathcal{E}$. (This is well defined because of the compatibility of $T(V^0, \leq, \preceq)$ and $\mathcal{G}$.)

For $(a, b) \in V'$, we define its *projection* $\pi : V \times V^0 \to V$ by putting $\pi((a, b)) = a$.

Definition

The tuple $(V^0, V^1, \preceq, \leq^0, \leq^1)$ is compatible with the 3-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$ iff:

- $V^0 \cap V^1 = \emptyset$,
- $(\leq^0, \preceq\restriction_{V^0})$ is a compatible pair of orders of V^0 and $T(V^0, \leq^0, \preceq\restriction_{V^0})$ is compatible with $\mathcal{G}$,
- $(\leq^1, \preceq\restriction_{V^1})$ is a compatible pair of orders of V^1 and $T(V^1, \leq^1, \preceq\restriction_{V^1})$ is compatible with the neighbourhood graph $G^1 = (V^1, E^1)$ of $\mathcal{G}$ with respect to $T(V^0, \leq^0, \preceq\restriction_{V^0})$,
- $\preceq$ is a well pre-order which satisfies $a \neq b$, $a \preceq b$, $b \preceq a \Rightarrow \pi(a) = \pi(b)$, and both projections are defined. Moreover, whenever $\pi(a)$ and $\pi(b)$ are defined, $\pi(a) \preceq \pi(b) \Rightarrow a \preceq b$. Finally, for $(a, b), (c, d) \in V^1$, we have $((a, b) \wedge (c, d)) \prec (b \wedge d)$.

Big Ramsey degrees of 3-uniform hypergraphs are finite

Theorem (Balko, Chodounský, H., Konečný 2019+)

Let H_3 be the random 3-uniform hypergraph.

$$\forall_{\textit{finite hypergraph } G, k\geq 1} H_3 \longrightarrow (H_3)^G_{k,T(G)}.$$

The big Ramsey degree of a 3-uniform hypergraph $G = ([n], \mathcal{E})$ in $\mathcal{H}_3$ is the number of non-isomorphic structures $([2n-1] \cup V^1, \preceq, \leq^0, \leq^1, \mathcal{E}, \mathcal{P})$ such that $([2n-1], V^1, \preceq, \leq^0, \leq^1)$ is compatible with $\mathcal{E}$, $\preceq\restriction_{[2n-1]}$ is a linear order and $\mathcal{P}$ consists of all triples $\{a, b, (a, b)\}$ such that (a, b) is a vertex of the neighbourhood graph G^1.

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Big Ramsey degree bounds are known for the following:

1. Order of integers (Ramsey 1930)
2. Order of rationals (Devlin 1979)
3. Random graph (Sauer 2006)
4. Dense local order (Laflamme, Nguyen Van Thé, Sauer 2010)
5. Ultrametric spaces (Nguyen Van Thé 2010)
6. Universal K_k-free graphs for $k \geq 2$ (Dobrinen 2018+)
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Kechris, Pestov and Todorcevic linked big Ramsey degrees to topological dynamics. This was recently developed by Andy Zucker to the notion of Big Ramsey structures.

Thank you for the attention

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