Big Ramsey degrees of the universal homogeneous parital order

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Big Ramsey Degrees

Theorem (Infinite Ramsey Theorem, 1930)

$$\forall_{p,k \geq 1} : \omega \longrightarrow (\omega)^{p}_{k,1}.$$

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$N \longrightarrow (n)^p_{k,t}$: For every partition of $\binom{\omega}{p}$ into k classes (colors) there exists $X \in \binom{\omega}{\omega}$ such that $\binom{X}{p}$ belongs to at most t parts.

($t = 1$ means that $\binom{X}{p}$ is monochromatic.)

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Structural formulation:

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Let $\mathcal{O}$ be the class of all finite linear orders.

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of **B** to **A**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colors.

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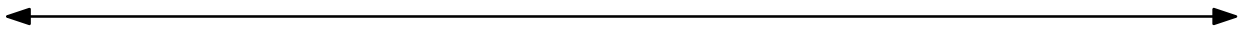
Sierpiski: not true for $|O| = 2$.

Rich coloring of $\mathbb{Q}$



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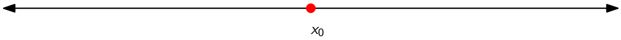

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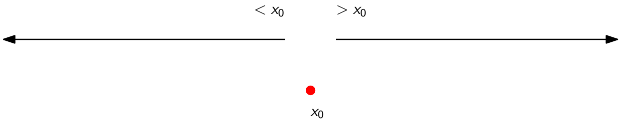

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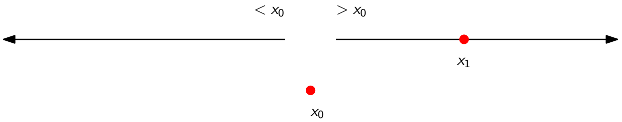

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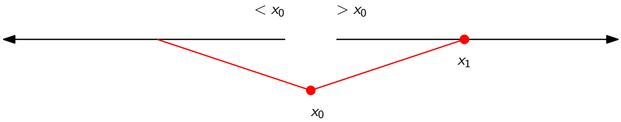

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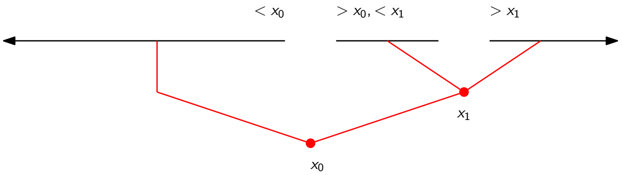

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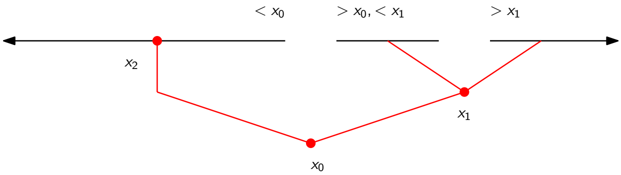

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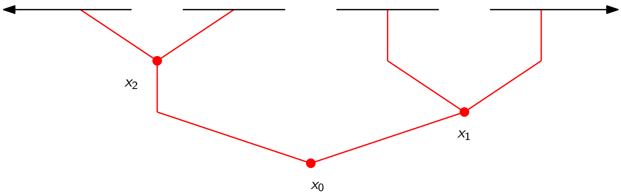

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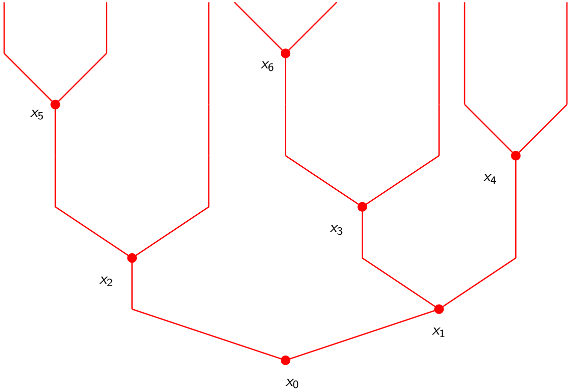

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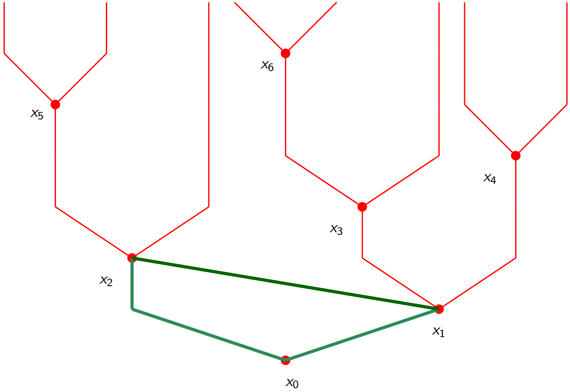

Color of k-tuple = shape of meet closure in the tree

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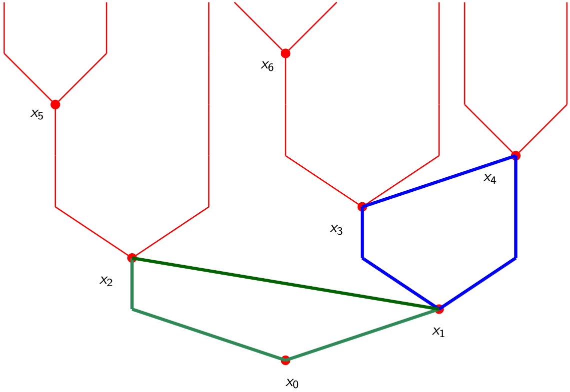

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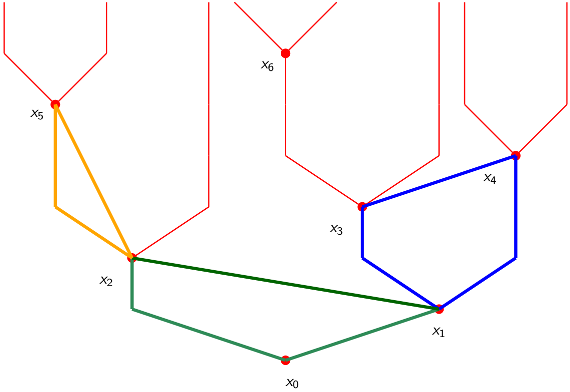

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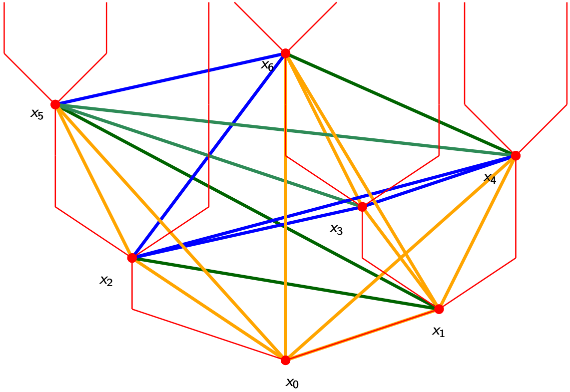

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Big Ramsey Degrees

In 1970's Laver developed method of finding copies of $\mathbb{Q}$ in $\mathbb{Q}$ with bounded number of colors using Milliken's tree theorem.

Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q},\leq) \longrightarrow (\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

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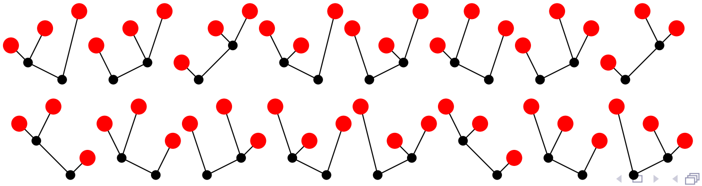

Other results on big Ramsey degrees

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7. Balko, Chodounský, H., Konečný, Vena (2020+): Big Ramsey degrees of 3-uniform hypergraphs are finite.
8. Dobrinen (2020+) and indepdendently by Balko, Chodounsk"y, H, Konečný, Vena, Zucker: Big Ramsey degrees of the homogeneous triangle-free graph.
9. Coulson, Dobrinen, Patel (2020+): Big Ramsey degrees of structures satisfying the Substructure Disjoint Amalgamation Property.

Main result

Let $\mathcal{P}$ be the class of all finite partial orders. By $(P, \leq)$ we denote the (countable) universal homogeneous partial order.

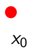
Theorem (H. 2020+)

The (countable) universal homogeneous partial order $(P, \leq)$ has finite big Ramsey degrees:

$$\forall_{(O,\leq)\in\mathcal{P}} \exists_{T=T(|O|)\in\omega} \forall_{k\geq 1} : (P, \leq) \longrightarrow (P, \leq)^{(O,\leq)}_{k,T}.$$

Universality: every countable partial order has embedding to $(\mathcal{P}, \leq)$.
Homogeneity: every partial isomorphism of two finite substructures of $(\mathcal{P}, \leq)$ extends to an automorphism.

Tree of types of $(P, \leq)$



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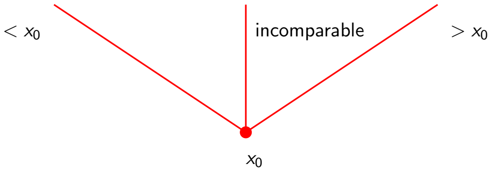

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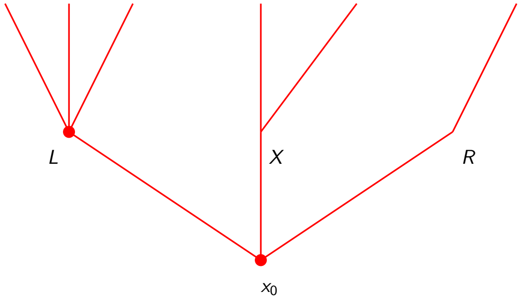

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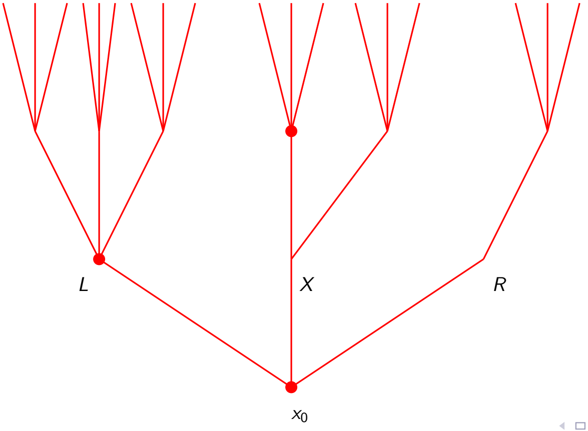

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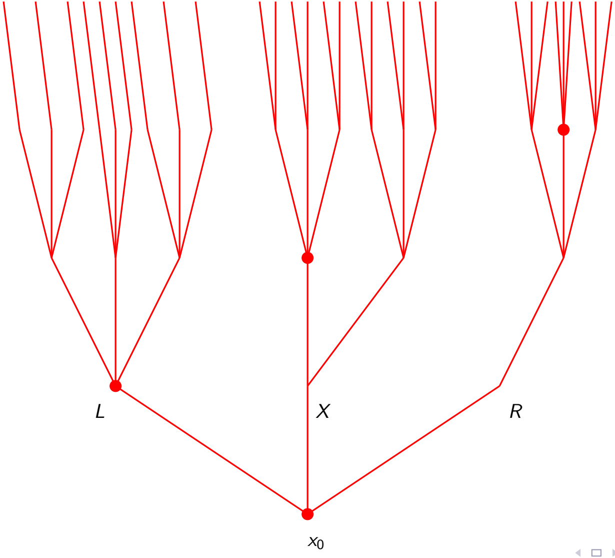

Definition (Parameter word)

Given a finite alphabet Σ and $k \in \omega + 1$, a *k-parameter word* is a (possibly infinite) string W in alphabet $\Sigma \cup \{\lambda_i \colon 0 \leq i < k\}$ containing each of λ_i, $0 \leq i < k$, such that for every $1 \leq j < k$, the first occurrence of λ_j appears after the first occurrence of λ_{j-1}.

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$\Sigma = \{L, X, R\}$.

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For set S of parameter words and a parameter word W:

$$W(S) = \{W(U) : U \in S\}.$$

Notation

We will denote the set of all finite k-parameter words of length at most n by:

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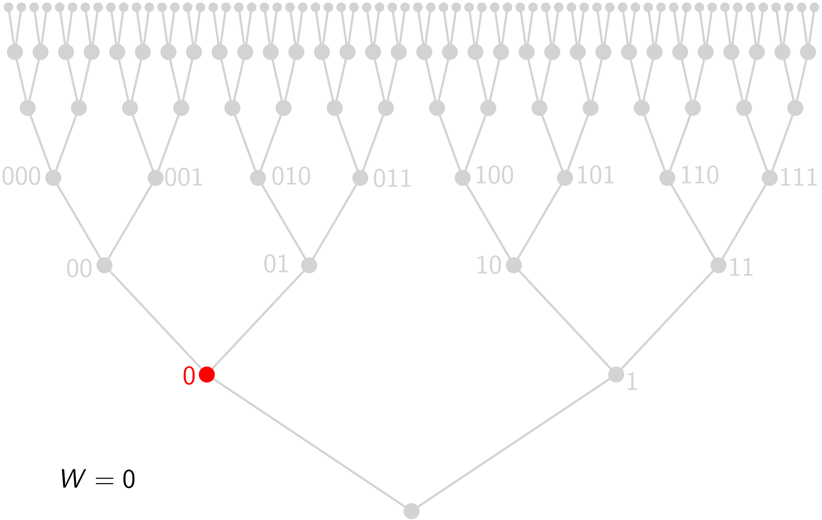
The following infinitary version of Graham–Rothschild Theorem is a direct consequence of the Carlson–Simpson theorem. It was also independently proved by Voight in 1983 (apparently unpublished):

Theorem

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set $[\Sigma]^* \binom{\omega}{k}$ is coloured by finitely many colours, then there exists an infinite-parameter word W such that $W\left([\Sigma]^* \binom{\omega}{k}\right)$ is monochromatic.

Parameter words as subtrees



$W = 0$

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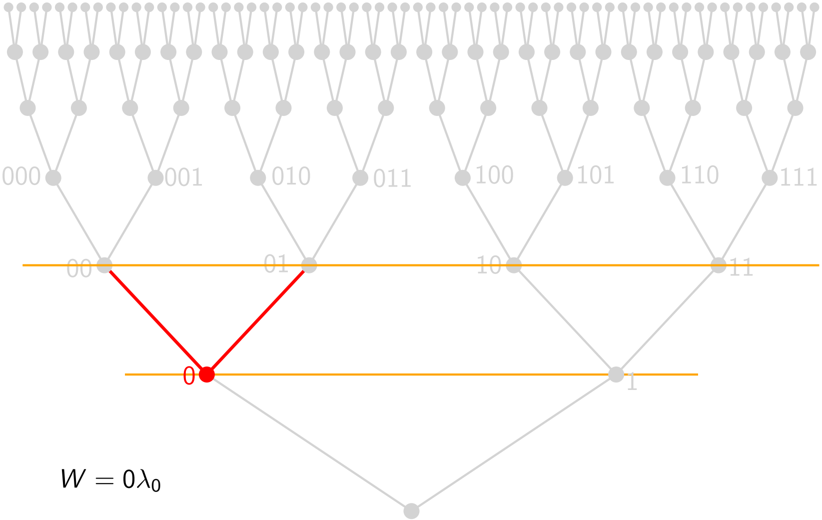

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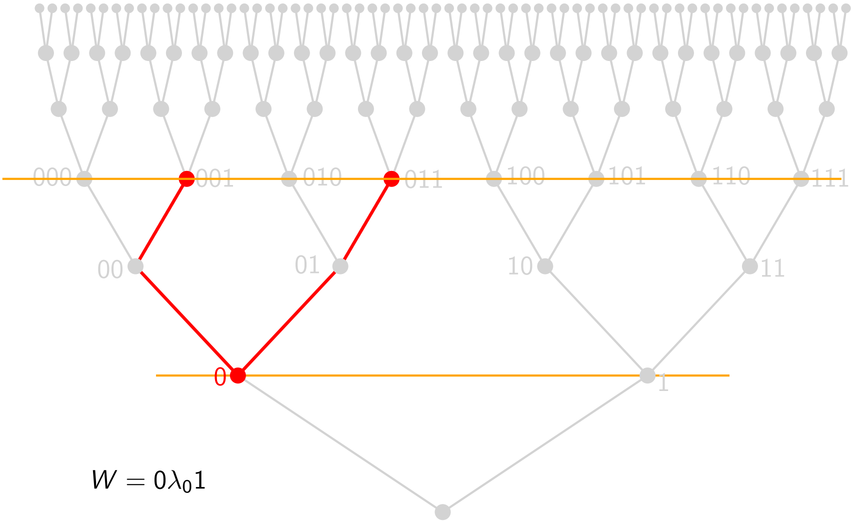

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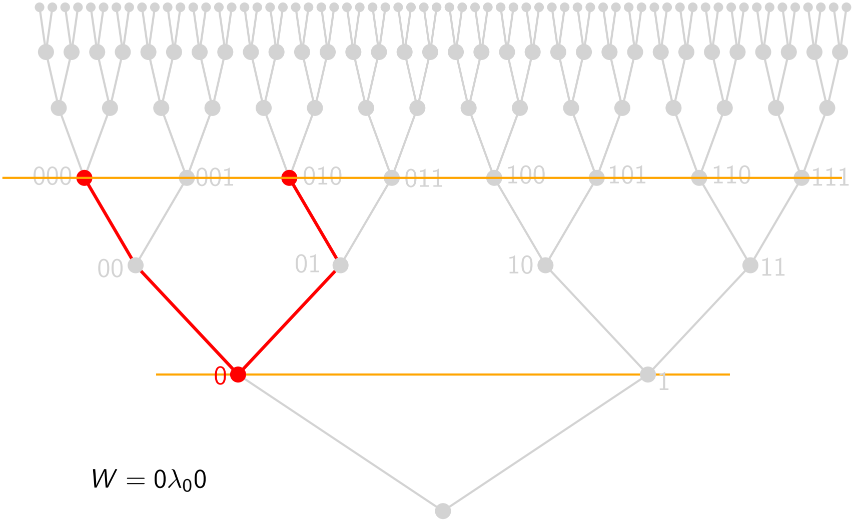

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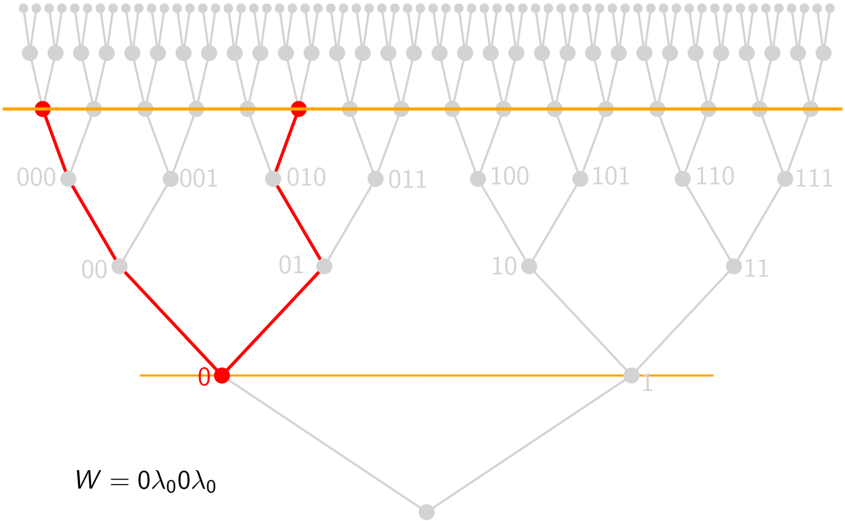

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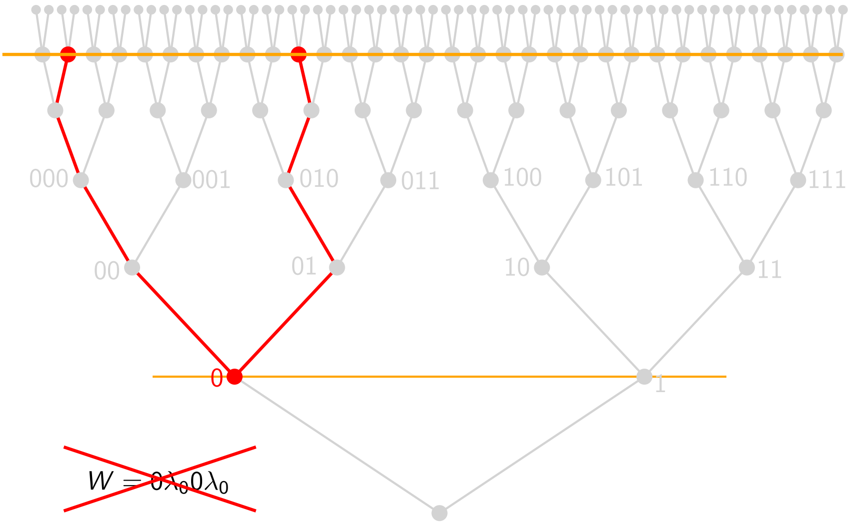

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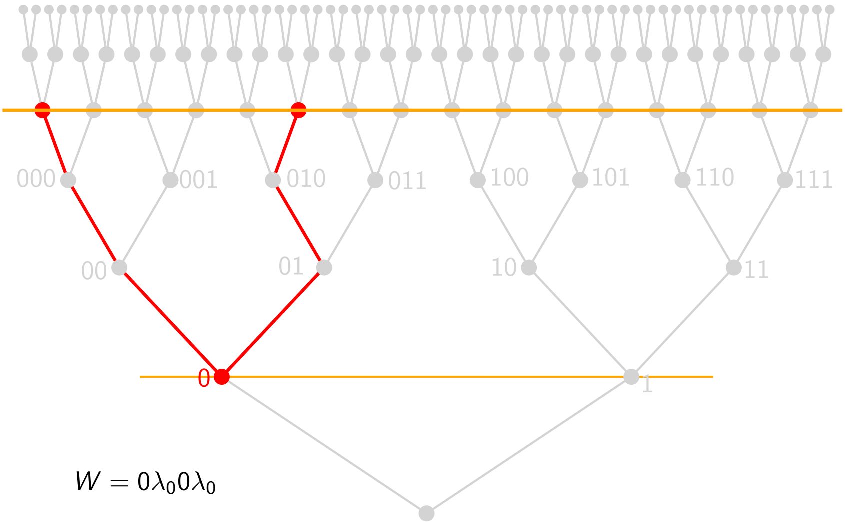

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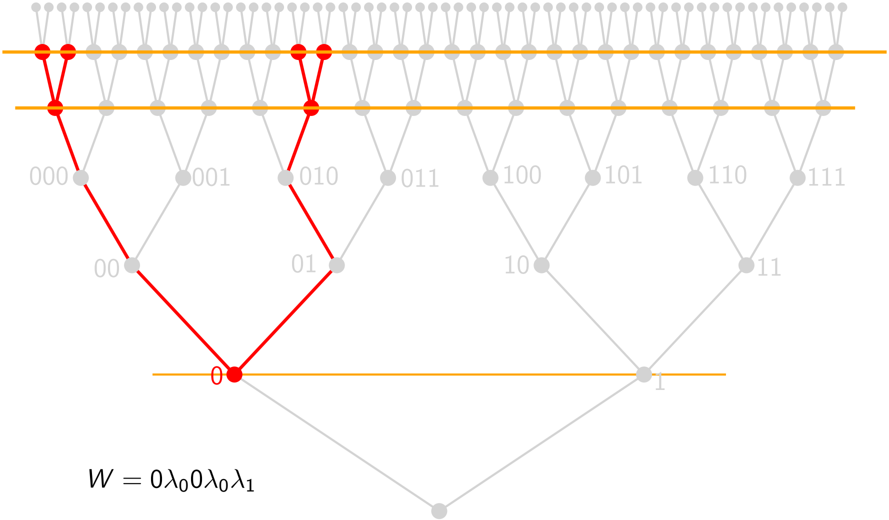

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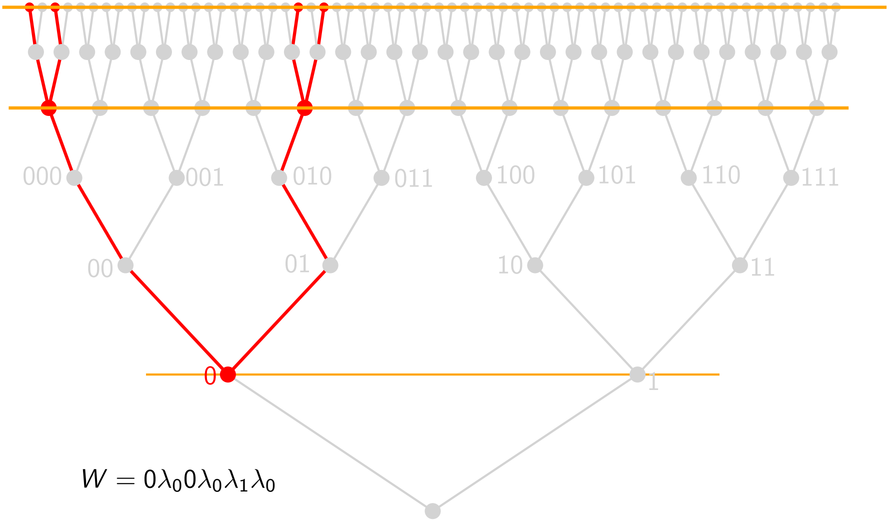

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Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set $S \subseteq [\Sigma]^* \binom{\omega}{k}$, an envelope of S is an n-parameter word W (for some $n \geq k$) satisfying $S \subseteq W([\Sigma]^* \binom{n}{k})$.
Envelope W is *minimal* if there is no envelope with fewer parameters.

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has two envelopes: $0\lambda_0\lambda_0$ and $0\lambda_0 0$. Thus $n = 1$.

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Proof.

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U	$=$	0	1	1	0	1	
V	$=$	0	0	1	1	0	1
Envelope:		0	λ_0				

Proposition

Let Σ be a finite alphabet, let $k \geq 0$ be a finite integer, let $S \subseteq [\Sigma]^* \binom{\omega}{k}$ be a finite non-empty set and let W be an envelope of S. Then W has at most

$$(|\Sigma| + k)^{|S|} + |S| - |\Sigma|$$

parameters.

Proof.

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Envelope:		0	λ_0	1	λ_1	λ_0	λ_2

□

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□

Definition

Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set $S \subseteq [\Sigma]^* \binom{\omega}{k}$ with an envelope W, an embedding type of S, denoted by $\tau(S)$, is the set of parameter words such that $W(\tau(S)) = S$.

$$\tau(U) = 10, \tau(V) = 011$$

Partial orders as words

$$\Sigma = \{L, X, R\}$$

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$$\Sigma = \{L, X, R\}$$
$$L <_{\text{lex}} X <_{\text{lex}} R$$

Definition

For $w, w' \in \Sigma^*$ we put $w \prec w'$ if and only if there exists $0 \leq i < \min(|w|, |w'|)$ such that

1. $(w_i, w'_i) = (L, R)$ and
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$u =$	L	X	X	L				
$v =$	L	R	X	R	X	L		
$w =$	X	R	R	R	X	R	X	X

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$$\begin{array}{llllllllll} u = & L & X & X & L & & & & \\ v = & L & R & X & R & X & L & & \\ w = & X & R & R & R & X & R & X & X \end{array}$$

$$u \preceq v, v \preceq w$$

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$$u \preceq v, v \preceq w \implies u \preceq w$$

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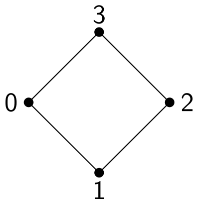
$(\Sigma^*, \preceq)$ is a universal partial order

$$\begin{aligned} u_0 &= L \quad R \\ u_1 &= \\ u_2 &= \\ u_3 &= \end{aligned}$$



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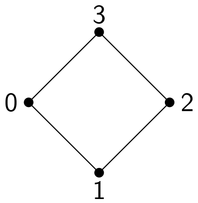

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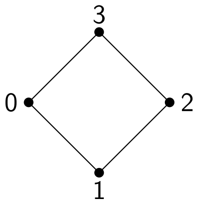
u_1 & = & L & L & L & R & & \\
u_2 & = & X & X & R & R & L & R \\
u_3 & = & & & & & &
\end{array}$$



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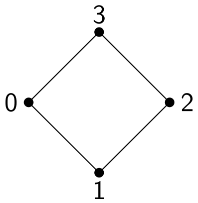

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Observation

For every infinite-parameter word W it holds that $u \preceq v \iff W(u) \preceq W(v)$.

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$W(v)$	$=$	X	R	R	R	X	R	X	X

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Theorem

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set $[\Sigma]^* \binom{\omega}{k}$ is coloured by finitely many colours, then there exists an infinite-parameter word W such that $W\left([\Sigma]^* \binom{\omega}{k}\right)$ is monochromatic.

Theorem (H. 2020+)

$\forall_{(O,\leq)\in\mathcal{P}} \exists_{T=T_p(|O|)\in\omega} \forall_{k\geq 1} : (P, \leq) \longrightarrow (P, \leq)^{(O,\leq)}_{k,T}$.

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Fix $(O, \leq)$ and a finite coloring of $\binom{(P,\leq)}{(O,\leq)}$. This gives a coloring of $\binom{(\Sigma^*,\preceq)}{(O,\leq)}$.

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Proof.

Fix $(O, \leq)$ and a finite coloring of $\binom{(P,\leq)}{(O,\leq)}$. This gives a coloring of $\binom{(\Sigma^*,\preceq)}{(O,\leq)}$. Because envelopes of copies of $(O, \leq)$ are bounded in dimension, apply the theorem above for every embedding type and obtain a copy of $(\Sigma^*, \preceq)$ with bounded number of colors.

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Fix $(O, \leq)$ and a finite coloring of $\binom{(P,\leq)}{(O,\leq)}$. This gives a coloring of $\binom{(\Sigma^*,\preceq)}{(O,\leq)}$. Because envelopes of copies of $(O, \leq)$ are bounded in dimension, apply the theorem above for every embedding type and obtain a copy of $(\Sigma^*, \preceq)$ with bounded number of colors. By universality the copy of $(\Sigma^*, \preceq)$ contains a copy of $(O, \leq)$. □

Theorem (M. Balko, D. Chodounský, J.H., M. Konečný, J. Neetil, L. Vena, 2020+)

Let L be a finite language consisting of binary relations only and $\mathcal{K}$ be an amalgamation class of L-structures satisfying:

1. *for every* $\mathbf{A} \in \mathcal{K}$*, every* $v \in \mathcal{K}$ *and every* $R \in L$ *it holds that* $(v, v) \notin R_{\mathbf{A}}$*,*

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2. *structure* **B** $\mathcal{K}$*-completion if and only if every induced cycle in* **B** *has* $\mathcal{K}$*-completion,*

Then the Fraïssé limit of $\mathcal{K}$ has finite big Ramsey degrees.

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Structure **A** is irreducible if for every $u, v \in A$, $u \neq v$ exists $R \in L$ such that (u, v) or (v, u) is in $R_{\mathbf{A}}$.
A is an (strong) completion of **B** if for every irreducible substructure **C** of **B** the identity is an embedding $\mathbf{B} \to \mathbf{A}$.

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Structure $\mathbf{A}$ is irreducible if for every $u, v \in A$, $u \neq v$ exists $R \in L$ such that (u, v) or (v, u) is in $R_{\mathbf{A}}$.
$\mathbf{A}$ is an (strong) completion of $\mathbf{B}$ if for every irreducible substructure $\mathbf{C}$ of $\mathbf{B}$ the identity is an embedding $\mathbf{B} \to \mathbf{A}$.

Example

1. Linear orders, partial orders
2. Triangle-free graphs
3. Ultrametric spaces with finitely many distances

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Example

1. Linear orders, partial orders
2. Triangle-free graphs
3. Ultrametric spaces with finitely many distances
4. Λ-ultrametric spaces with finitely many distances

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Example

1. Linear orders, partial orders
2. Triangle-free graphs
3. Ultrametric spaces with finitely many distances
4. Λ-ultrametric spaces with finitely many distances
5. Metric spaces with distances $\{0, 1, \ldots, d\}$

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Example

1. Linear orders, partial orders
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3. Ultrametric spaces with finitely many distances
4. Λ-ultrametric spaces with finitely many distances
5. Metric spaces with distances $\{0, 1, \ldots, d\}$
6. S-metric spaces, S finite satisfying the 4-values condition.

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5. Metric spaces with distances $\{0, 1, \ldots, d\}$
6. S-metric spaces, S finite satisfying the 4-values condition.
7. Metric spaces associated to 3-constrained metrically homogeneous graphs from Cherlin's list

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2. *structure $\mathbf{B}$ $\mathcal{K}$-completion if and only if every induced cycle in $\mathbf{B}$ has $\mathcal{K}$-completion,*

Then the Fraïssé limit of $\mathcal{K}$ has finite big Ramsey degrees.

Structure $\mathbf{A}$ is irreducible if for every $u, v \in A$, $u \neq v$ exists $R \in L$ such that (u, v) or (v, u) is in $R_{\mathbf{A}}$.
$\mathbf{A}$ is an (strong) completion of $\mathbf{B}$ if for every irreducible substructure $\mathbf{C}$ of $\mathbf{B}$ the identity is an embedding $\mathbf{B} \to \mathbf{A}$.

Example

1. Linear orders, partial orders
2. Triangle-free graphs
3. Ultrametric spaces with finitely many distances
4. Λ-ultrametric spaces with finitely many distances
5. Metric spaces with distances $\{0, 1, \ldots, d\}$
6. S-metric spaces, S finite satisfying the 4-values condition.
7. Metric spaces associated to 3-constrained metrically homogeneous graphs from Cherlin's list
8. Free superpositions of the above.

Tree of types

Lower bounds can be proved by generalizing techniques of Laflamme, Sauer and Vuksanovic (2006).

1. Fix your favorite infinite structure **A**.
2. Fix enumeration: assume that its vertex set is ω
3. Determine types: Given time $t \in \omega$ and vertices $u, v \in A$, $u, v > t$ put $u \simeq_t v$ if u and v extends the structure induced by **A** on $\{1, 2, \ldots t\}$ the same way.
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6. Determine unavoidable shapes: A shape is unavoidable if subtree induced by every copy of **A** in **A** contains the shape.

What are the shapes

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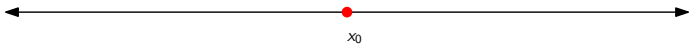

$$\mathit{Vertex}(t_0 \to x_0), \mathit{Branch}(t_0 \to t_1, t_2),$$

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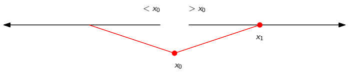

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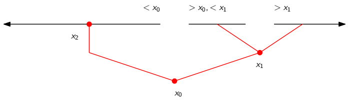

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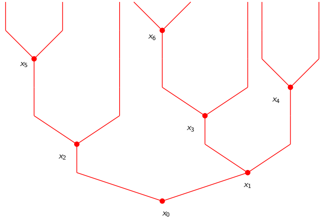

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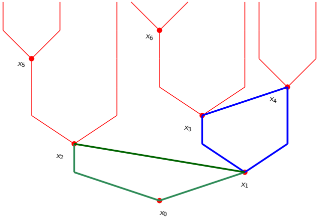

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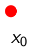
Tree of types of $(P, \leq)$



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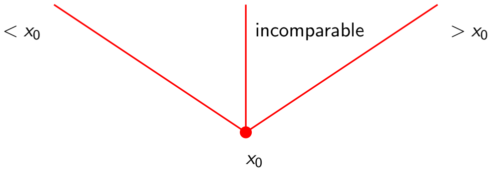

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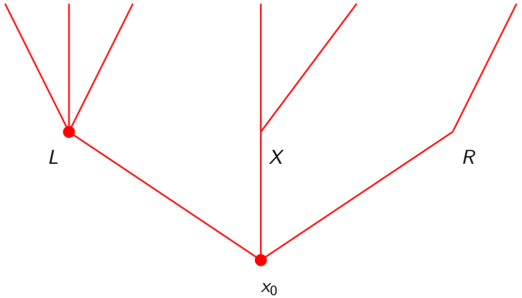

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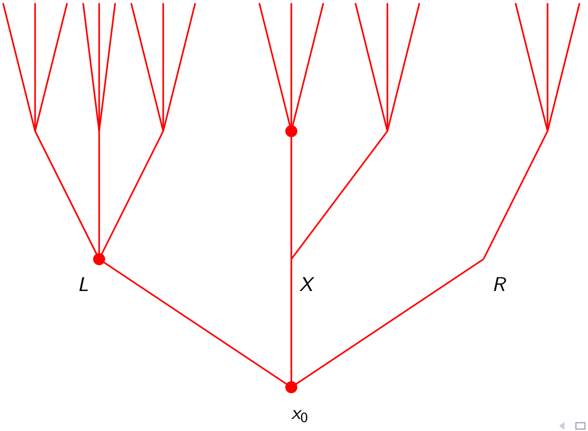

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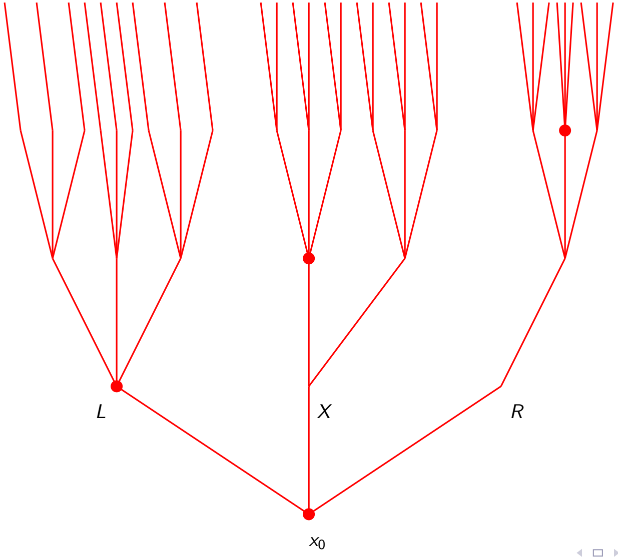

Tree of types of the countable random graph

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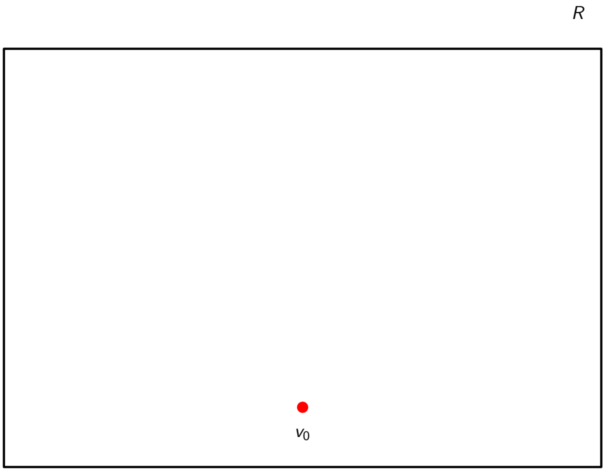

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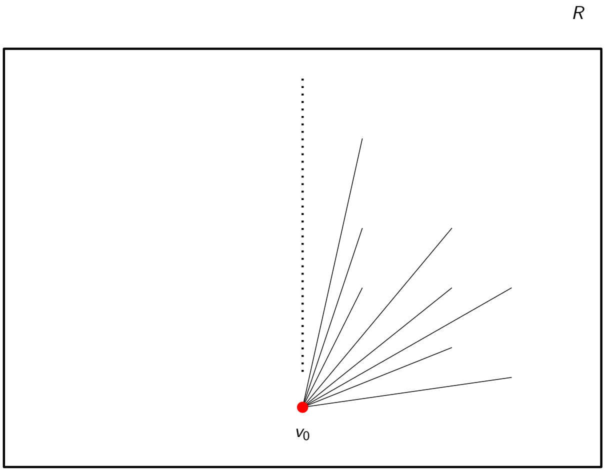

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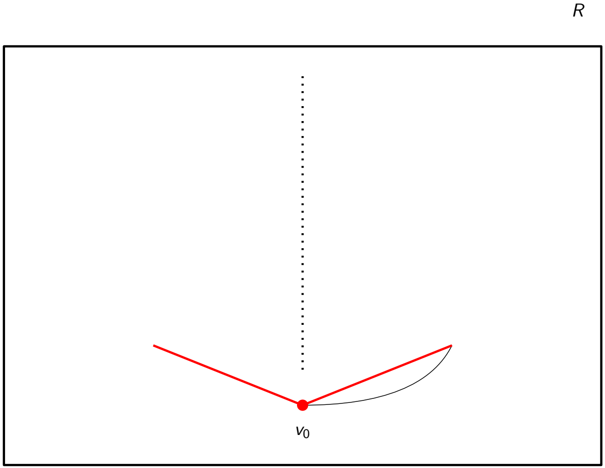

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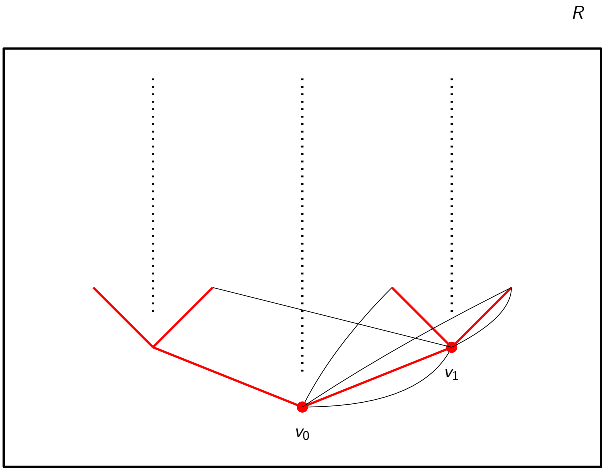

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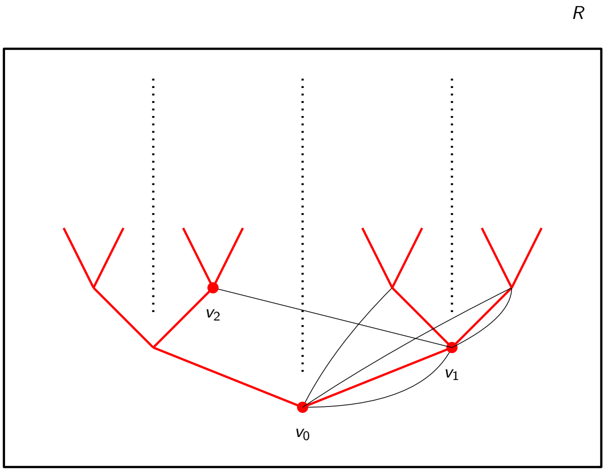

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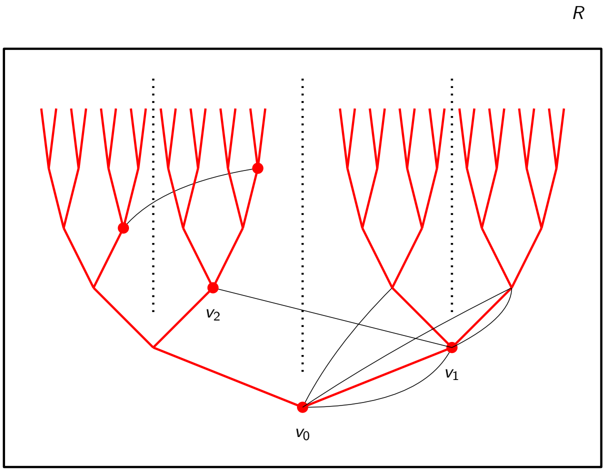

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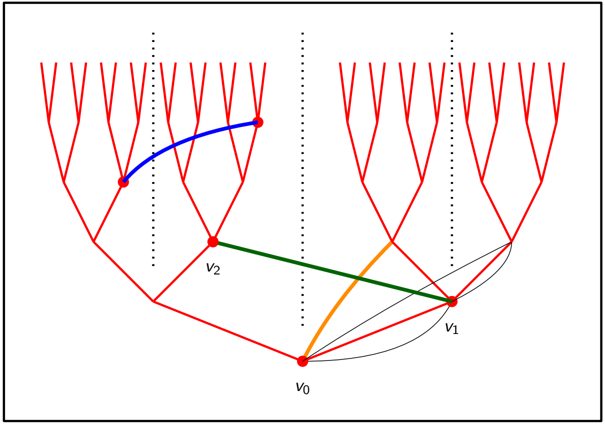
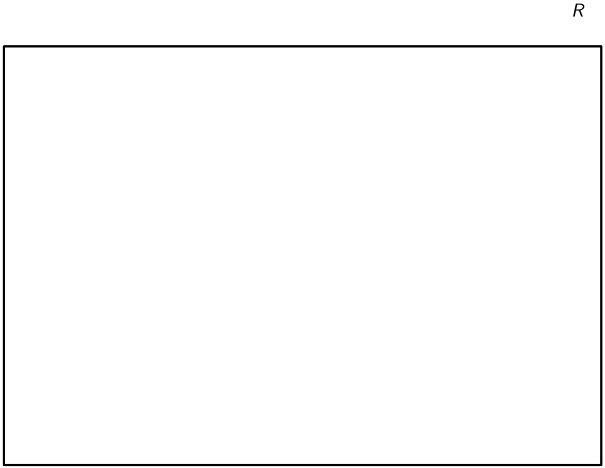


Tree of types of the countable generic triangle-free graph



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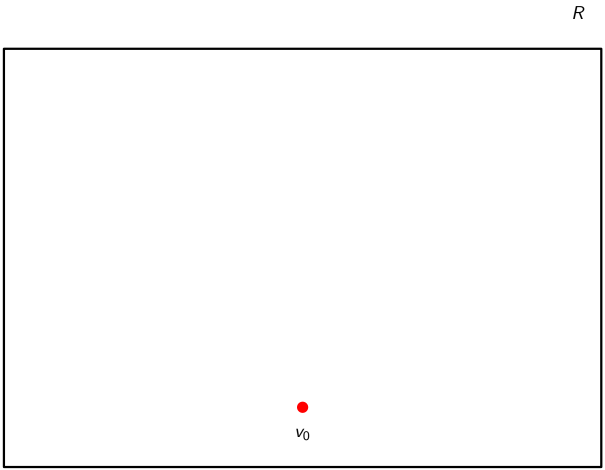

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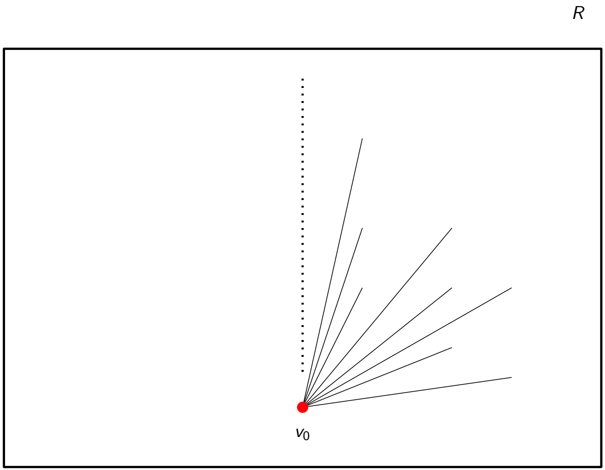

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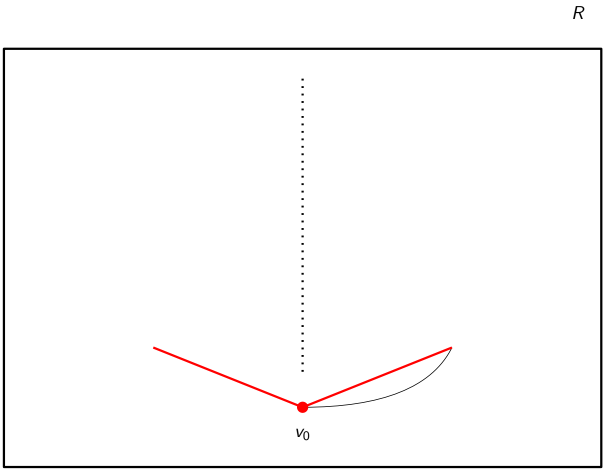

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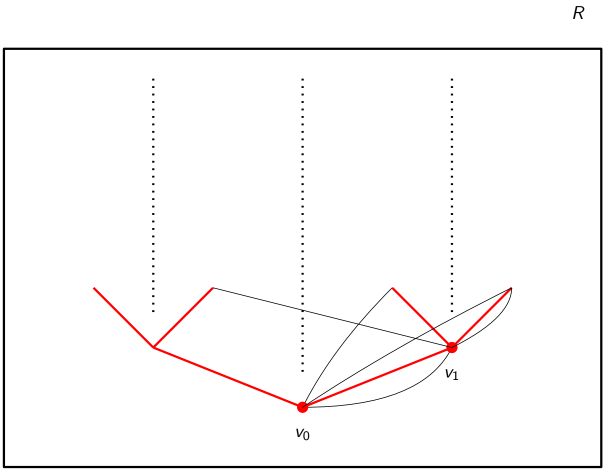

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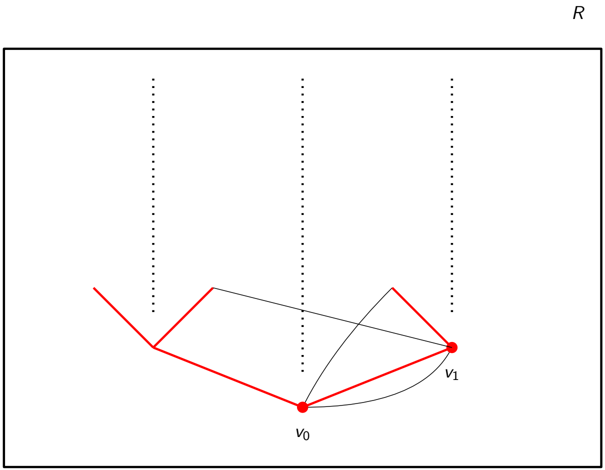

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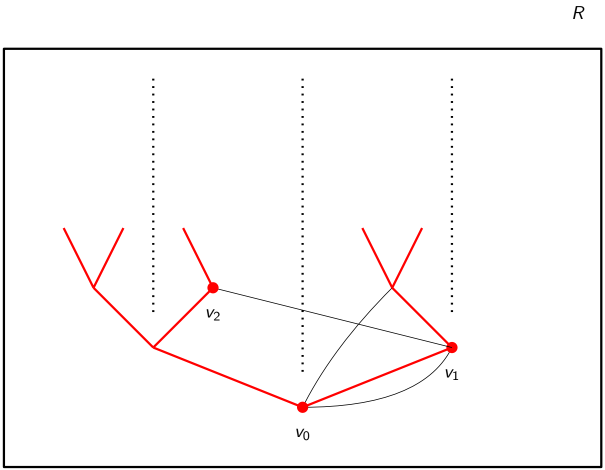

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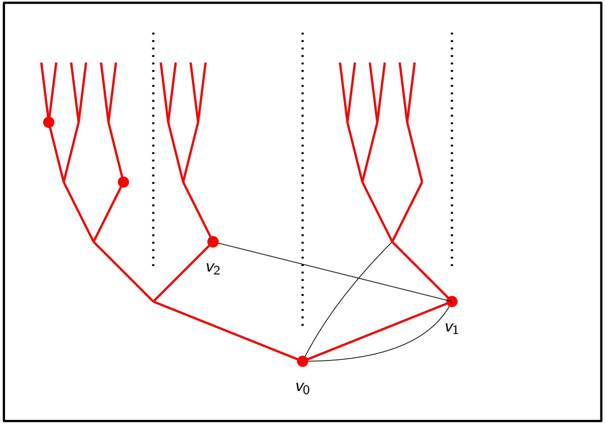
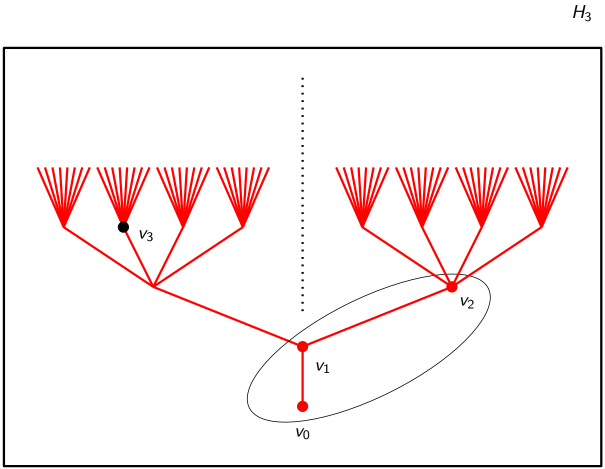


Tree of types of the countable random 3-uniform hyper-graph



Color of a subgraph = shape of meet closure in the tree

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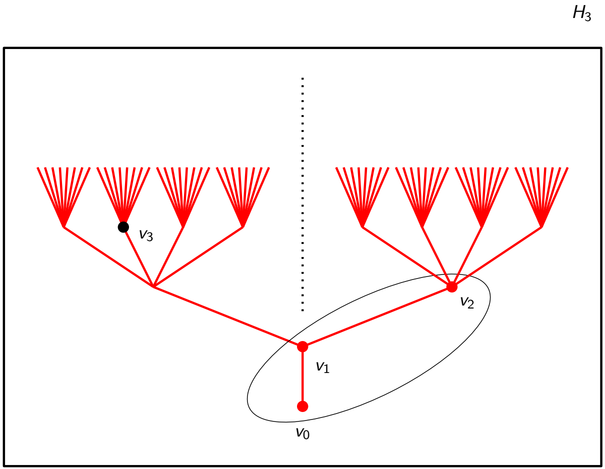

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Year later we observed that neighborhood of a vertex is the Random graph!

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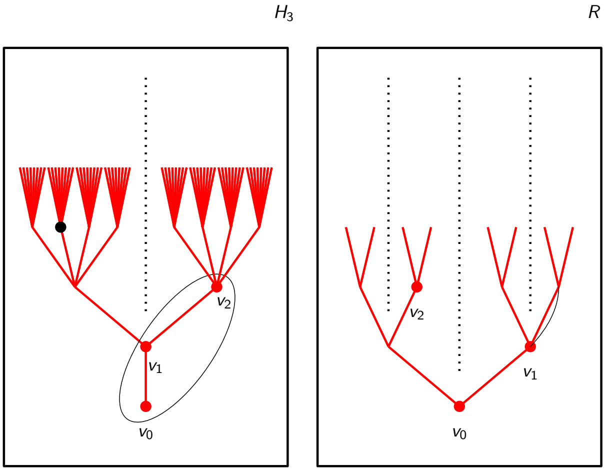

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Thank you for the attention

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