EPPA for two-graphs

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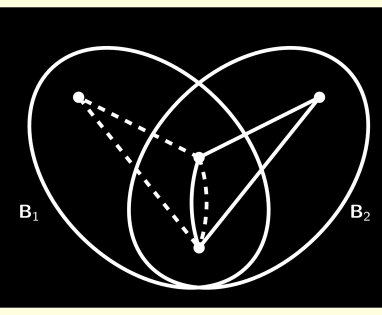
Joint work with Jaroslav Nešetřil and David Evans
On the occasion of Dougald Macpherson's 60th birtday

Does the class of all finite two-graphs have the extension property for partial automorphisms (EPPA)?

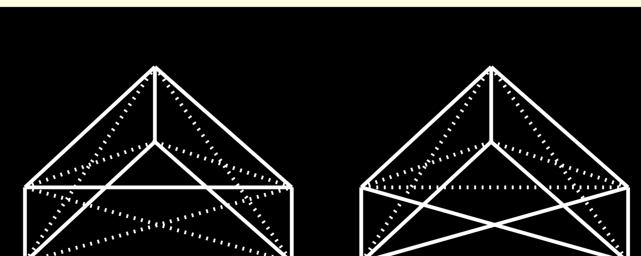
Aha, two-graphs are interestig animals. I will tell you three definitions.

Two-graphs are 3-uniform hypergraphs such that on every 4 distinct vertices there is an even number of hyper-edges.

Equivalently, every two-graph can be created from a graph by putting a hyper-edge on every triple of vertices containing an odd number of edges.



I see, for every three edges of label 3, the induced edges of label 1 form either two triangles or a six-cycle. The two-graph's vertices are the edges of label 3 and the hyper-edges correspond to the six-cycles. I heard about it on Peter Cameron's talk. He calls it a clique double-cover.

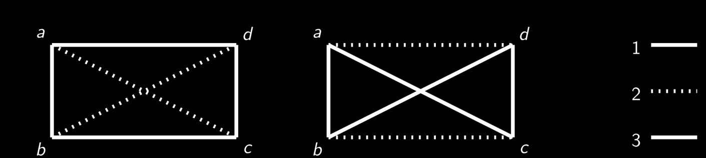


Recall!
Class of structures $\mathcal{K}$ has EPPA if for every $\mathbf{A} \in \mathcal{K}$ there exist $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A} \subseteq \mathbf{B}$ and every isomorphism between two substructures of $\mathbf{A}$ (partial automorphism of $\mathbf{A}$) extends to an automorphism of $\mathbf{B}$

Every two-graph corresponds to an antipodal metric space. Antipodal metric spaces have only distances 1, 2 and 3 such that

- edges of distances 3 form a perfect matching
- there are no triangles with distances 1, 1, 3 and 2,2,3.

It is practical to consider metric spaces as special edge-labelled graphs (complete graphs labelled by the distances).



Here is an interesting question:
Two-graphs can not be described by a finite family of forbidden homomorphisms and hence the Herwig–Lascar theorem does not apply.

I have no direct proof of it, but having such family would also imply Ramsey property of ordered two-graphs. By the KPT-correspondence we know that every Ramsey expansion of two-graphs fixes a graph. Most likely this will be true for EPPA too. Let's ask a group theorist about it!

I can think of a completion problem both for two-graphs and antipodal metric spaces that has no automorphism preserving solution. This is just like tournaments. EPPA for tournaments is a well known open problem.

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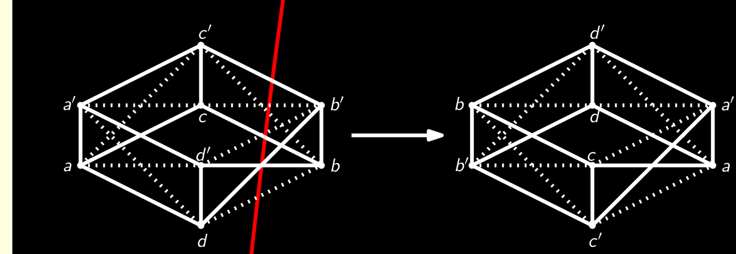
Photos of fictional mathematicians courtesy of Šechtl & Voseček Museum of Photography

After two years of hard work!

Let's try to prove EPPA for antipodal metric spaces. I don't know how to do this. I can show that for every antipodal metric space $\mathbf{A}$ there exists an edge-labelled graph $\mathbf{B}$ which extends all partial automorphisms of $\mathbf{A}$ and moreover the edges with label 3 form a matching. Let me write it on the blackboard.

Fix an arbitrary finite metric space $\mathbf{A}$ with ℓ antipodal pairs. Let O be the set of all edges of $\mathbf{A}$ of length 1. Observe that $|O| = 2\binom{\ell}{2}$ and put $m = \ell - 1$. We now construct a $\{1,2,3\}$-edge-labelled graph $\mathbf{B}$ as follows.

1. Vertices are pairs (X, X') of disjoint subsets of O of size m.
2. We connect (X, X') and (Y, Y') by an edge according to the following conditions:
   (a) edge of length 1 if $|X \cap Y| = 1$, $|X' \cap Y'| = 1$, $X \cap Y' = \emptyset$ and $X' \cap Y = \emptyset$,
   (b) edge of length 2 if $X \cap Y = \emptyset$, $X' \cap Y' = \emptyset$, $|X \cap Y'| = 1$ and $|X' \cap Y| = 1$,
   (c) edge of length 3 if $X = Y'$ and $X' = Y$
   (d) no edge otherwise.



OK, it easilly follows that edges with label 3 form a perfect matching.

One can find a copy $\mathbf{A}'$ of $\mathbf{A}$ in $\mathbf{B}$ by assigning to each $a \in A$ the pair consiting of the set of all edges of length 1 containing a and the set of all edges of length 1 containing a' which is the unique vertex in distance 3 from a.

Every partial automorphism of $\mathbf{A}'$ corresponds to a partial automorhism of $\mathbf{A}$ and a partial permutation of O. Extend the permutation to get an automorphism of $\mathbf{B}$. This is similar to an argument in the Herwig–Lascar paper and dates back to Erdős.

I recently saw a cool construction of Hodkinson–Otto. It can be modified to solve our problem. Let me write the rest of the construction for you.
I will write $d_{\mathbf{B}}(x, y)$ for the label of the edge connecting vertices x and y in $\mathbf{B}$ if it exists.

Given vertex $x \in B$, denote by $U_{\mathbf{B}}(x)$ the set of all vertices $y \in B \setminus \{x\}$ such that there is no edge connecting x and y.
We construct $\mathbf{C}$ as follows:

1. Vertices of $\mathbf{C}$ are pairs (x, χ_x) where $x \in B$ and χ_x is a function from $U_{\mathbf{B}}(x)$ to $\{0,1\}$ such that for every pair $y, y' \in U_{\mathbf{B}}(x)$ satisfying $d_{\mathbf{B}}(y, y') = 3$ it holds that $\chi_x(y) = \chi_x(y')$.
2. Distances are given by the following rules:
   (a) $d_{\mathbf{C}}((x,\chi_x),(y,\chi_y)) = d_{\mathbf{B}}(x,y)$ if $x \neq y$ is connected by an edge in $\mathbf{B}$ and $d_{\mathbf{B}}(x,y) < 3$,
   (b) $d_{\mathbf{C}}((x,\chi_x),(y,\chi_y)) = 1$ if $x = y$,
   (c) $d_{\mathbf{C}}((x,\chi_x),(y,\chi_y)) = 1$ if $x \neq y$, there is no edge connecting x and y and $\chi_x(y) = \chi_y(x)$,
   (d) $d_{\mathbf{C}}((x,\chi_x),(y,\chi_y)) = 2$ if $x \neq y$, there is no edge connecting x and y and $\chi_x(y) \neq \chi_y(x)$,
   (e) $d_{\mathbf{C}}((x,\chi_x),(y,\chi_y)) = 3$ if $d_{\mathbf{B}}(x,y) = 3$ and $\chi_x(z) = 1 - \chi_y(z)$ for every $z \in U_{\mathbf{B}}(x)$.
   (f) $d_{\mathbf{C}}((x,\chi_x),(y,\chi_y)) = 2$ if $d_{\mathbf{B}}(x,y) = 3$ and rule 2e does not apply.

Is $\mathbf{C}$ is an antipodal metric space? Will it extend partial automorhisms??
Read our paper or ask us!
D. Evans, J. Hubička, M. Konečný, J. Nešetřil: EPPA for two-graphs